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Techniques for Capturing Critical Thinking in the Creation and Composition of Advanced Mathematical Knowledge

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1. Introduction

1.1 Research into Advanced Mathematical Behaviour

Advanced mathematical thinking (Tall, 1991) and tertiary level mathematics education research (Selden & Selden, 2002) have only recently become established research fields. Research into the working practices of mathematicians is still rare. In a recent article, I observed that

> unlike most other subjects, mathematical activity resides almost entirely within the cognitive processes of a mathematics practitioner and is therefore difficult to characterise. Despite recent interest, the nature of advanced mathematical activity remains something of a *black box* to educational researchers (Samuels, 2012, p. 1).

Apart from major mathematical discoveries, such as Wiles' experience of proving Fermat's Last Theorem (Singh, 1997), mathematicians' rich and profound experiences of doing advanced mathematics have generally lacked a language and vehicle of expression. In approximately the last 150 years, the discourse of the mathematics research community has focused almost entirely upon the product of mathematical activity rather than its process (Science Festival Foundation, 2013; Solomon & O'Neill, 1998), leading me to express my sense of *alienation* from the product of my mathematical labour (Samuels, 1993). Assuming I am not alone, I hope that the data capturing techniques presented in this paper will provide mathematicians with a variety of means to share what they are thinking as they create and communicate advanced mathematics.

1.2 Purpose, Perspective and Outline

The purpose of this paper is to present new techniques for capturing critical thinking in the process of creating and writing up advanced mathematics. The aim is to complement, rather than challenge, the standard, product-orientated genre of academic mathematical discourse. The proposed techniques presented here are based neither on the standard data capturing techniques used in previous research into mathematical behaviour nor on a requirement that mathematicians have the additional identity and capability of being researchers in mathematical behaviour. Furthermore, these techniques do not assume that the research will be initiated by mathematical behavioural researchers observing mathematicians and deriving insight into their thinking processes from these observations which have an inherent risk of being invalid (which will be discussed later). Instead, they provide a means for mathematicians to capture and communicate rich data into their actual working practices.

Four techniques are introduced with examples from my own research into analytical fluid mechanics: plan writing, concept mapping, activity transcripts, and annotated drafts and transcripts. Each of these techniques is fairly easy to use and unobtrusive as they do not involve another researcher being present, or capturing data in a potentially distracting manner, or mathematicians spending additional time participating in contrived activities outside of their

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normal working practices. They also cover different stages in the process of creating mathematics and composing mathematical writing, as discussed below.

Given that I am a research mathematician, and one of the goals of this paper is to promote a division of labour between research mathematicians and researchers in mathematical behaviour, I have not attempted to analyse my own critical thinking from my mathematical data as this would contradict this division of labour. It would also create the additional problems of a lack of objectivity and a dual identity, setting an unhelpful precedent which I do not wish others necessarily to follow. The absence of analysis of the critical thinking in the examples of the proposed techniques provided might be viewed as a weakness of the paper in validating their merits relative to existing techniques. However, a more general evaluation of the proposed techniques is provided in Sections 3 and 4.

As a concession to this possible perceived weakness, the examples of the proposed techniques have been selected because they appear to contain critical thinking and provide different perspectives on the process of creating and writing up the same piece of advanced mathematics which other behavioural researchers may wish to analyse further. The examples are therefore provided more for the purpose of promoting the creation of a corpus of mathematical process data and encouraging future analysis, as discussed in Section 6, rather than being of direct interest to the average *Double Helix* reader.

This paper builds on the ideas I presented in a recent opinion piece (Samuels, 2012). In Section 2, the issue of critical thinking in science and mathematics is explored. In Section 3, existing techniques for capturing data on advanced mathematical behaviour are critiqued. In Section 4, in order to provide a framework for discussing these techniques, the relationship between the process of creating mathematics and the writing process is explored. Each proposed technique is then presented in turn in Section 5 with examples from my doctoral research into analytical fluid mechanics (Samuels, 2000). Finally, in Section 6, these proposed techniques are compared with existing techniques used by mathematical behavioural researchers, their utility is evaluated, and the possibility of creating a corpus of similar behavioural data is discussed.

2. Critical Thinking in Science and Mathematics

The development of critical thinking is widely accepted as being important within academia, but there is considerable disagreement over its definition. In an extensive study of university academic staffs' views on the subject, Paul et al. (1997) found that "few have had any in-depth exposure to the research on the concept and most have only a vague understanding of what it is and what is involved in bringing it successfully into instruction." Moon (2008) argued for a definition which emphasises utility to learners. Her literature review identified a variety of approaches: some, such as Gillett (2014), defined critical thinking as the application of Bloom's (1956) taxonomy (understanding, analysis, synthesis and evaluation) to an area of knowledge; others, such as Fisher (2001), emphasised the application of logic to critiques and arguments; others, such as Cottrell (2011), viewed critical thinking in terms of a collection of component skills; others have taken an overview perspective. Of these overview perspectives, perhaps the best recognised is that of Ennis (1989) who defined critical thinking as "reasonable and reflective thinking focused on deciding what to believe or do" (p. 4).

Ennis (1989) also characterised different views on whether critical thinking differs according to the subject area to which it is applied, leading to different implications for the way it should be taught. Firstly, the *epistemological subject specificity* view holds that good thinking has different forms in different subject areas. The National Council for Excellence in Critical Thinking

(2013) appears to adhere to this view, having stated that

> instruction in all subject domains should result in the progressive disciplining of the mind with respect to the capacity and disposition to think critically within that domain. Hence, instruction in science should lead to disciplined scientific thinking; instruction in mathematics should lead to disciplined mathematical thinking; …and in a parallel manner in every discipline and domain of learning.

Secondly, the *conceptual subject specificity* view argues that generic critical thinking is impossible because thinking is always applied to something. Bailin (2002) supported this view within the context of science education, encouraging its application through "focusing on the tasks, problems and issues in the science curriculum which require or prompt critical thinking" (p. 370). However, common to both these views is the requirement to understand the nature of knowledge within a discipline before critical thinking within it can be understood.

The nature of mathematical knowledge can be seen as a special case of scientific knowledge due to mathematics' position as "queen and servant of the sciences" (Bell, 1951): queen in the sense of being the abstraction of the concepts, objects and procedures used in other areas of science, and servant in the sense that all science disciplines use mathematics to present knowledge. There is considerable debate amongst philosophers on the nature of scientific knowledge (Eflin et al., 1999), which includes issues such as the unity of science, the demarcation of science from other subjects and whether scientific paradigms are consistent or contradictory. Regarding the nature of learning activities, Pask (1976) differentiated physical sciences from the arts and social sciences. He defined the former as *operational style*, which Ramsden (1997) summarised as "the manipulation of concepts and objects within the subject-matter domain, the emphasis on procedure-building, rules, methods, and details" (p. 209). Pask defined the latter as *comprehension style*, which Ramsden (1997) summarised as "the description and interpretation of the relations between topics in a more general way" (p. 209). His differentiation implies there is much less scope for analysing, evaluating and interpreting ideas within physical sciences.

In general terms, there are fundamental distinctions between a mathematical assertion that is universally accepted being true, a formal argument demonstrating that it is true and a reader of such an argument both intuitively "seeing" it is true and being convinced it is true by the argument provided. A simple example is Pythagoras' Theorem, which is universally accepted as true but a proof is seldom provided (see http://www.mathscentre.ac.uk/video/1090/ for an intuitive argument).

The nature of mathematical knowledge has been the subject of extensive philosophical debate for over a hundred years. Its foundation is largely attributed to Frege (Kitcher & Aspray, 1988). He also led the debate from which the three main positions for viewing mathematical knowledge were established: *logicism*, which views mathematics as a logical system, the main work being Whitehead and Russell's *Principia Mathematica* (1910-1913); *formalism*, which views mathematics in terms of provably consistent formal systems, the main protagonist being Hilbert (1926), and which led to the *Bourbaki Programme* of standard exposition of mathematics (Mashaal, 2006); and *intutionism*, developed by Brouwer (1948), which asserts that the fundamental properties of mathematical objects should be based on intuition rather than logic. According to Kitcher and Aspray (1988), these three main positions still dominate the argument today.

However, each of these positions shares the belief that mathematical knowledge is a *formal* *system of deduction* whose axioms and rules can be precisely stated and followed. One construct is built upon another with formal proofs provided for any assertions. Results presented are either true or false and should be critically evaluated in these absolute, objective terms of validity (Goldin, 2003). Two famous examples are Russell's letter to Frege just before his major work on mathematical foundations (Frege, 1903) went to press, which completely undermined it by identifying a logical flaw in his argument, known as *Russell's Paradox* (Hersh, 1997, p. 148), and Wiles' proof of Fermat's Last Theorem (Singh, 1997), which was held up for over a year by a technical difficulty due to one minor oversight in his original (incorrect) proof.

Furthermore, there are additional forms of critical thinking in mathematics apart from the formal validation of mathematical arguments. Schoenfeld (1992) emphasised the need to develop effective mathematical thinking in the context of problem solving and metacognition. His approach aligns closely with the epistemological subject specificity view and the "deciding what to do [next]" (p. 4) aspect of Ennis' (1989) definition of critical thinking. Schoenfeld (1992, p. 356) reported an experiment in which he compared the ability of college and high school students with that of staff mathematicians in solving non-standard problems. He found the latter spent much more time in analysis, exploration and planning, leading to much higher success levels from which he concluded that staff mathematicians were more adept at mathematical thinking in this context.

The focus of critical thinking in this article is on its use in the creation of advanced mathematical knowledge. From the epistemological subject specificity view, the main recognised work on critical thinking in this area is by the Advanced Mathematical Thinking Working Group of the International Group for the Psychology of Mathematics (Tall, 1991). In particular, in agreement with the observation made above, Tall (1991) recognised the importance of precise definitions and logical proof in advanced mathematical thinking, noting that "the move from elementary to advanced mathematical thinking involves a significant transition: that from describing to defining, from convincing to proving in a logical manner based on these definitions" (p. 20). Furthermore, consistent with the example of Pythagoras' Theorem above, Dreyfus (1991) stressed the importance of being able to move between an intuitive understanding of an assertion and a formal proof that it is true. The purpose of this paper is to present techniques which have the potential to shed light on what mathematicians are thinking as they create and write up advanced mathematics.

3. Evaluation of Existing Data Capture Techniques

There are major problems with the use of traditional behavioural research techniques to capture data concerning advanced mathematical behaviour. Nardi et al.'s (2005) observational study of undergraduate mathematics tutorials is perhaps the most relevant, although the level of mathematics is slightly lower than that discussed in this paper. Observations are, however, time-consuming to analyse and the completed analysis may not reflect what the students were actually thinking at the time, especially if they contributed little verbally, since most mathematical creative activity takes place in silence.

Other studies into the behaviour of working mathematicians have involved researchers conducting interviews (Burton, 2001) and focus groups (Iannone & Nardi, 2005) with mathematicians analysing mathematical texts (Burton & Morgan, 2000), video recordings of mathematical problem solving behaviour (Schoenfeld, 1985) or mathematicians providing personal reflections into their own behaviour (Poincaré, 1908). However, the use of each of these approaches for capturing advanced mathematical behaviour is problematic: most rely on mathematicians providing rationalisations of past behaviour which are subject to criticism of post-rationalisation and dissonance from thinking during the activity (Nisbett & Wilson, 1977). Schoenfeld’s (1985) video study of the mathematical problem solving process is very insightful, but this technique is not applicable to capturing advanced mathematical behaviour. Burton and Morgan’s (2000) textual analysis was applied to completed texts, representing the product of mathematical behaviour, rather than its process. In summary, these techniques are either not applicable to capturing the behaviour of research mathematicians or inappropriate for capturing their processes of creating and writing up advanced mathematics—see Table 1.

Table 1 *Comparison of Existing Behavioural Research Techniques for Investigating Advanced Mathematical Behaviour*

Data capturing technique	Example	Applicable to research mathematicians	Captures the mathematical creative process	Captures the mathematical writing process
Observation	(Nardi et al., 2005)	No	No	No
Interview	(Burton, 2001)	Yes	No	No
Focus group	(Iannone & Nardi, 2005)	Yes	No	No
Textual analysis	(Burton & Morgan, 2000)	Yes	No	No
Video analysis	(Schoenfeld, 1992)	No	Possibly	No
Reflection	(Poincaré, 1908)	Yes	Not in detail	No

The possibility of an alternative approach appears to be difficult. The complexity of analysing mathematical behavioural data provided by interviews and textual analysis, and the underlying complexity of the phenomena they describe, may have discouraged researchers in mathematics behaviour from seeking to obtain more authentic data due to the belief that the analysis of such data might be even more resource intensive and complex. For example, the direct observation of mathematicians doing mathematics would be intrusive and might require a long period of time. Another underlying assumption is that research into the working practices of mathematicians must be initiated by researchers into mathematics behaviour; mathematicians are generally treated as research subjects according to the classical positivist research paradigm.

Iannone and Nardi’s (2005) co-researcher approach is an exception. They adopted an interpretive paradigm, treating mathematicians more equally by exploring the conditions under which mutually effective collaboration between mathematicians, such as those they enlisted, and researchers in mathematics education, such as themselves, might be achieved. However, their use of prepared data sets and focus groups is very different from the one proposed here. On the whole, researchers in mathematical behaviour initiate research studies and generally consider using only the data capturing techniques with which they are familiar from other contexts.

One possible solution would be for research mathematicians to carry out ethnographic studies into their own behaviour. However, very few research mathematicians have either the capability or the interest to carry out an objective analysis into their own research processes. Such an approach has been described by Anderson (2006) as *analytical autoethnography*, in which the researcher is “a full member in the research group or setting, visible as such a member in the researcher’s published texts” (p. 375) (in this case, the mathematics research community) and “committed to an analytic research agenda focused on improving theoretical understandings of broader social phenomena" (p. 375) (in this case, the mathematical behaviour research community).

Two examples of autoethnographic studies are Tall's (1980) account and reflections of his discovery in infinitesimal calculus and Chick's (1998) application of the *Structure of the Observed Learning Outcome taxonomy* (Biggs & Collis, 1982) to her doctoral research in abstract algebra. Whilst both studies provide interesting insights into the process of creating mathematical knowledge, the lack of other similar or follow-on studies in the last 35 years illustrates the difficulty and rarity of this combined identity approach. The single identity approach of a mathematician as a *transcript provider* is easier for mathematicians to achieve and provides more detailed data. Therefore, it has a greater potential to provide more data of a richer quality, enabling researchers in mathematics behaviour to gain greater insight into the thought processes of mathematicians as they create mathematics.

4. The Mathematical Creative Process and the Writing Process

4.1 Process Models

Poincaré (1908) proposed a four stage model of mathematical creativity based on introspections on his own mathematical behaviour: *preparation*—conscious work on a problem, *incubation*—unconscious work, *illumination*—a sudden *gestalt* insight, and *verification*—another phase of conscious work to shape the insight (hereafter, his model is referred to as Poincaré's Gestalt Model, as a gestalt insight is its distinctive feature). At the time, mathematicians disagreed with Poincare's approach, as it was seen as a departure from rigour, leading in part to the Bourbaki Programme; however, this view is no longer mainstream (Senechal, 1998).

Poincaré's model is now widely accepted as the starting point for describing the creative process in general (Lubart, 2001). Hadamard's (1945) reflections on mathematical creativity are in close agreement with Poincaré's, whereas Ervynck (1991) suggested a three-stage model: a preliminary technical stage; algorithmic activity; and creative (conceptual, constructive) activity. However, a recent detailed study of the working practices of mathematicians by Sriraman (2004) showed strong agreement with Poincaré's Gestalt Model and Hadamard rather than Ervynck's model. Therefore Poincaré's Gestalt Model is adopted within this paper.

The writing process has also been characterised by a model containing sub-processes. Based on a literature review of previous studies, Humes (1983) proposed four such sub-processes: *planning*—generating and organising content and setting goals; *translating*—transforming meaning from thought into words; *reviewing*—looking back to assess whether what has been written captures the original sense intended; and *revising*—in which the writer can do anything from changing his/her mind, leading to major reformulations, to making minor edits to his/her text. These sub-processes are generally enacted in the order given here but can overlap and be revisited later during the writing process, as illustrated in Figure 1. As Humes' (1983) model is widely accepted, it has also been adopted within this paper (and is referred to hereafter as Humes' Sub-processes Model).

4.2 Interrelationship

Of the limited research into the relationship between the creation of mathematics and the creation of mathematical texts, perhaps most significant is that by Solomon and O'Neill (1998), who explored the relationship between mathematics and writing by considering the historical approach taken by mathematicians when the academic writing style was not dominant within the discourse

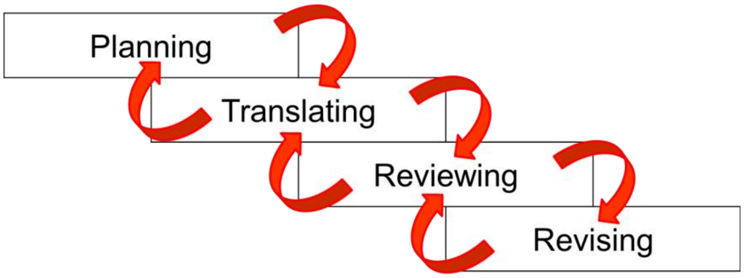


Figure 1. Humes' Sub-processes Model of writing composition.

of the professional mathematical community. In particular, they investigated the writing style used by Hamilton (1843) in his discovery of quaternions, reporting how he demonstrated fluency in switching between an informal narrative style and a formal journalistic style when communicating his findings in the appropriate social or institutional context. They argued for the importance of teaching a correct mathematical writing style rather than a reliance solely upon narrative genres for those who may feel excluded from the dominant mathematical discourse. However, a more important conclusion from their research for the current study is that the narrative writing style has almost entirely been lost by mathematicians due to the dominance of the standard, product-orientated mathematical style in the contemporary academic discourse, to the detriment of research into the working behaviour of mathematicians.

The approach taken by most authors of books on mathematical writing agrees with Solomon and O'Neill's (1998) recommendation to teach a correct mathematical writing style. For example, Vivaldi (2013) emphasised how to produce correct content according to the mathematical writing style. In addition, some authors provide limited contextualised advice on the mathematical writing process (Maurer, 2010). However, Aitchison and Lee (2006) dispute the adequacy of an emphasis solely on the mechanics of writing to account for the complexities of doctoral students' writing, let alone the writing by professional researchers. Therefore, there remain underlying tensions among advice on a formal mathematical writing style for communicating results, writing process models to improve mathematical writing and a narrative style for communicating the mathematical process.

Despite these unresolved tensions, a number of observations can still be made into the connection between Poincaré's Gestalt Model of mathematical creativity and Humes' Sub-processes Model of writing composition. Firstly, at least since the early Nineteenth Century (Caranfa, 2006), writing has been seen as a creative process. Therefore, due to the accepted general applicability of Poincaré's Gestalt Model, it would be expected that all stages of this model be present within the writing composition process to some extent.

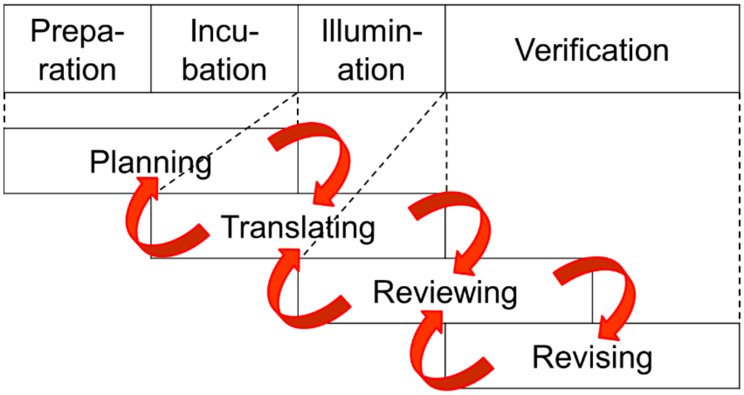


Figure 2. Similarities between Pointcaré's Gestalt Model and Humes' Sub-processes Model when applied to writing composition.

Secondly, Crowley (1977) observed similarities between some of the stages of Poincaré's Gestalt Model and the sub-processes of Humes' model: preparation and incubation are similar to planning; illumination is similar to translating; and verification is similar to revising and reviewing—see Figure 2. However, writing at the verification stage of the mathematical creative process is more for personal understanding than for planned communication with the mathematical community. Only if this activity has been successful and the mathematician decides it is sufficiently important to be communicated to the wider community will a second phase of translating (this time of the mathematical writing) be required.

Thirdly, and for the same reason as the second point above, the writing itself cannot usually be planned until the mathematical discovery has been completed, verified and reflected upon. Perhaps the most famous example of this is Wiles' communication of his proof of Fermat's Last Theorem (Singh, 1997), comprising his original lectures at Cambridge University; the slight problem he identified with his own argument; his subsequent over-coming of this problem and his publishing of a mathematical paper communicating his verified findings (Wiles, 1995). Therefore, in most circumstances, the stages of the mathematical creative process follow the sub-processes of the writing process. Figure 3 maps the four data capturing techniques proposed in this paper onto the mathematical creative process and the mathematical writing process. Table 2 provides more information on this comparison. These techniques will now be introduced and explored in turn through examples from my own doctoral research (Samuels, 2000). As already stated, the purpose of presenting these examples is to illustrate the techniques, rather than to analyse the meaning or significance of their content. However, they have been chosen carefully to exemplify potentially interesting critical thinking.

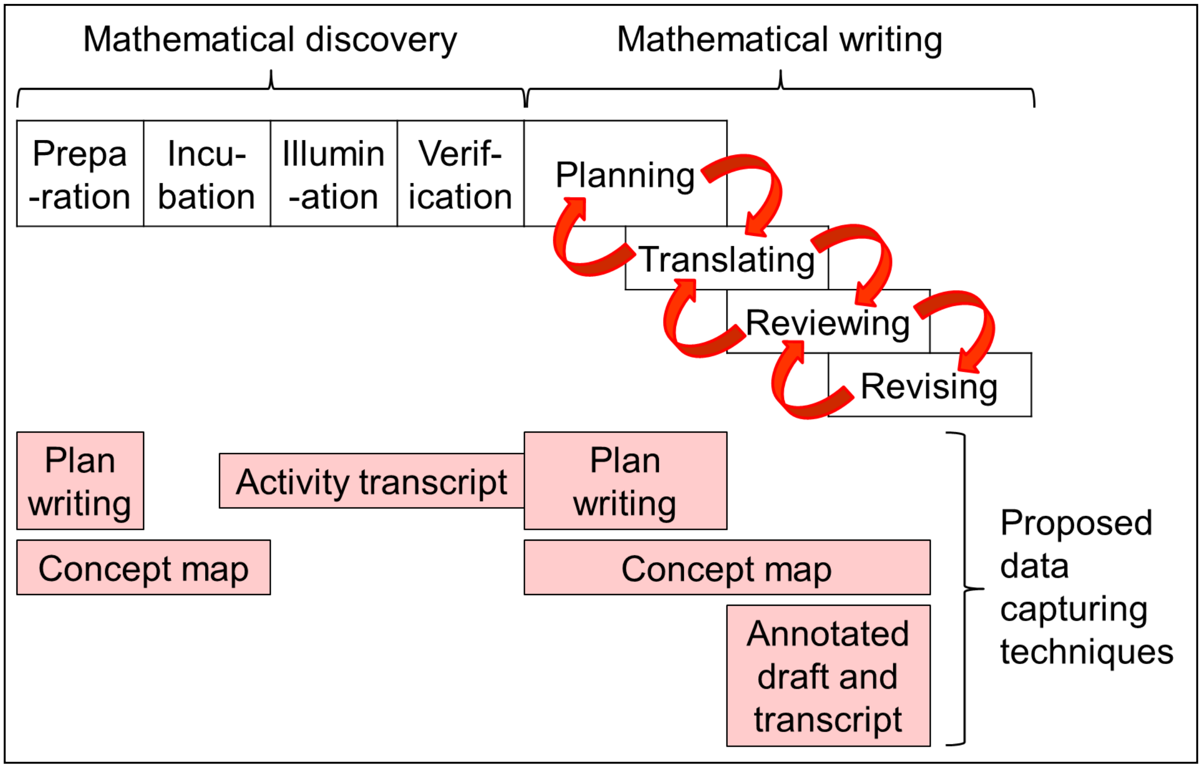


Figure 3. Mapping of proposed data capturing techniques onto the mathematical creativity and writing process.

Table 2 *Applicability of Proposed Data Capturing Techniques to the Mathematical Creative and Writing Processes*

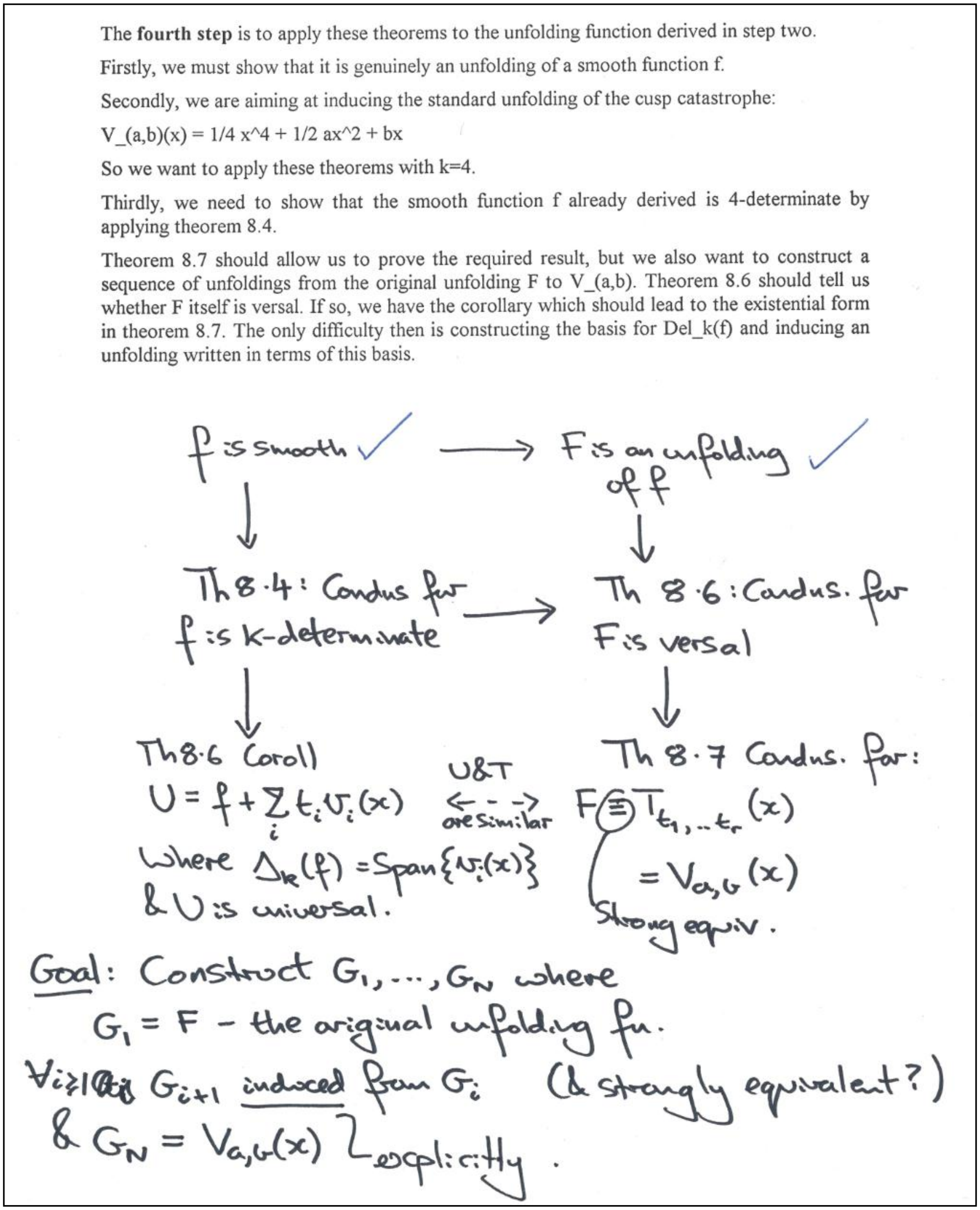
Technique	Static or dynamic	Corresponding stage of the mathematical creative process	Corresponding sub-process(es) of the mathematical writing process	Reference(s) to similar work
Plan writing	Static	Preparation	Planning	(Pólya, 1945; Pugalee, 2001)
Activity transcript	Dynamic	Notes made during activity could be written during preparation or verification	Account of activity similar to translating but in a narrative style	(Craig, 2011; Tall, 1980)
Concept map	Static	Any, especially preparation and incubation	Any, especially planning and reviewing	(Bolte, 1999; Kaufman, 2012; Lavigne et al., 2008; Mac Lane, 1986; Ojima, 2006)
Annotated draft and transcript	Dynamic	Preparation	Reviewing	(Eliot, 1971)

5. Data Capturing Techniques

5.1 Plan Writing

Plan writing is used here to describe a data capturing technique by which a mathematician elaborates on a plan to create a certain mathematical result. An example is provided in Figure 4. The printed text formed part of a communication to my supervisor in which I provided him with an overview of my plan to create a particular proof of a result on the application of catastrophe theory (Poston & Stewart, 1978) to nonlinear wave theory (Whitham, 1974). The handwritten notes were for my own benefit after I met with my supervisor. The other pages of this communication are provided in Appendix A. This plan relates more to creating the mathematical content. Figure 5 provides an overview plan of the same process which I produced for my own benefit. It relates to both the mathematical creativity process (Level 1) and the mathematical composition process (Level 2). Figures 4 and 5 illustrate how different forms of plan are created for different purposes. Plan writing relates to the preparation stage in the mathematical creative process and the planning stage in the mathematical composition process. It is a static technique in the sense that it captures current thinking rather than changes in thinking.

Very little has been written about capturing written mathematical plans as a data capturing technique. Pólya (1945) viewed planning as a vital step in mathematical problem solving. His description of this process is similar to the first stage in Poincaré's Gestalt model of mathematical creativity. Pugalee (2001) used written mathematical plans as a technique to investigate Year 9 students' metacognition in mathematical problem solving. However, neither of these authors nor those who have built on their work, such as Schoenfeld (1985), appears to have promoted plan writing as a technique for mathematicians to communicate their advanced mathematical behaviour.



The **fourth step** is to apply these theorems to the unfolding function derived in step two.

Firstly, we must show that it is genuinely an unfolding of a smooth function f.

Secondly, we are aiming at inducing the standard unfolding of the cusp catastrophe:

V_(a,b)(x) = 1/4 x^4 + 1/2 ax^2 + bx

So we want to apply these theorems with k=4.

Thirdly, we need to show that the smooth function f already derived is 4-determinate by applying theorem 8.4.

Theorem 8.7 should allow us to prove the required result, but we also want to construct a sequence of unfoldings from the original unfolding F to V_(a,b). Theorem 8.6 should tell us whether F itself is versal. If so, we have the corollary which should lead to the existential form in theorem 8.7. The only difficulty then is constructing the basis for Del_k(f) and inducing an unfolding written in terms of this basis.

f is smooth ✓ ⟶ F is an unfolding of f ✓

↓ ↓

Th 8.4: Condns for f is k-determinate ⟶ Th 8.6: Condns. for F is versal

↓ ↓

Th 8.6 Coroll

$U = f + \sum_i t_i v_i(x)$

Where $\Delta_k(f) = \text{Span}\{v_i(x)\}$

& U is universal.

U&T $\longleftarrow - - \longrightarrow$ are similar

Th 8.7 Condns. for:

$F \boxed{\equiv} T_{t_1,\dots t_r}(x)$

$= V_{a,b}(x)$

Strong equiv.

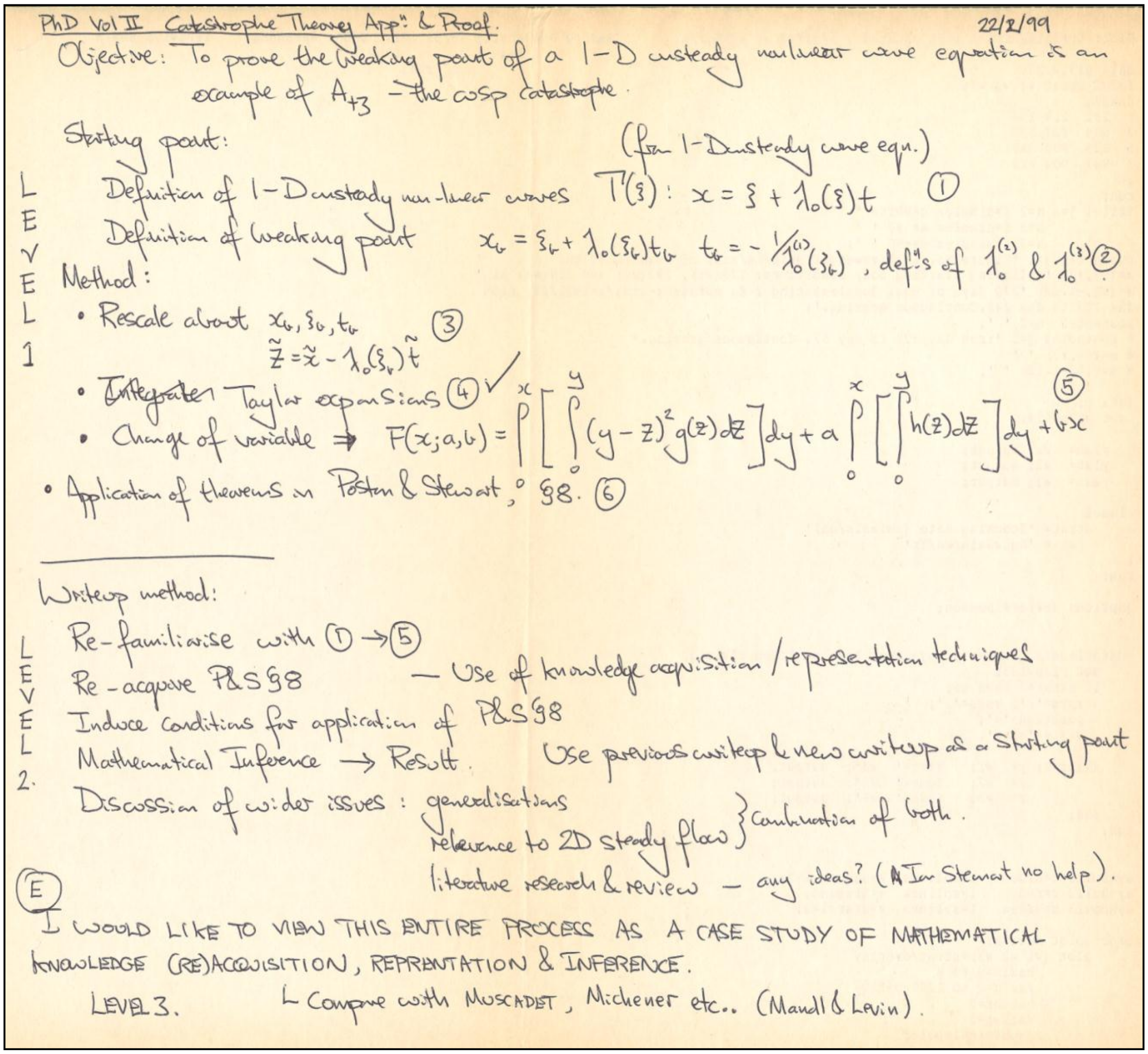
<u>Goal</u>: Construct $G_1, \dots, G_N$ where

$G_1 = F$ – the original unfolding fn.

$\forall i \geq 1$ G_{i+1} <u>induced</u> from G_i (& strongly equivalent?)

& $G_N = V_{a,b}(x)$ explicitly.

Figure 4. Example of plan writing.



PhD Vol II Catastrophe Theory Appⁿ & Proof. 22/2/99

Objective: To prove the breaking point of a 1-D unsteady nonlinear wave equation is an example of A_{+3} – the cusp catastrophe.

Starting point: (for 1-D unsteady wave eqn.)

L Definition of 1-D unsteady non-linear waves $\Gamma(\xi): x = \xi + \lambda_0(\xi)t$ ①

E Definition of breaking point $x_b = \xi_b + \lambda_0(\xi_b)t_b \quad t_b = -1/\lambda_0^{(1)}(\xi_b)$, defⁿs of $\lambda_0^{(2)}$ & $\lambda_0^{(3)}$ ②

V

E Method:

L
- Rescale about x_b, ξ_b, t_b ③

1 $\tilde{z} = \tilde{x} - \lambda_0(\xi_b)\tilde{t}$

- ~~Integrate~~ Taylor expansions ④ ✓
- Change of variable ⇒ $F(x;a,b) = \int_0^x \left[\int_0^y (y-z)^2 g(z)\,dz \right] dy + a \int_0^x \left[\int_0^y h(z)\,dz \right] dy + bx$ ⑤
- Application of theorems in Poston & Stewart, §8. ⑥

Writeup method:

L Re-familiarise with ① → ⑤

E Re-acquire P&S §8 — Use of knowledge acquisition / representation techniques

V Induce conditions for application of P&S §8

E

L Mathematical Inference → Result. Use previous writeup & new writeup as a starting point

2. Discussion of wider issues : generalisations

relevance to 2D steady flow } combination of both.

literature research & review – any ideas? (~~A~~ Ian Stewart no help).

Ⓔ

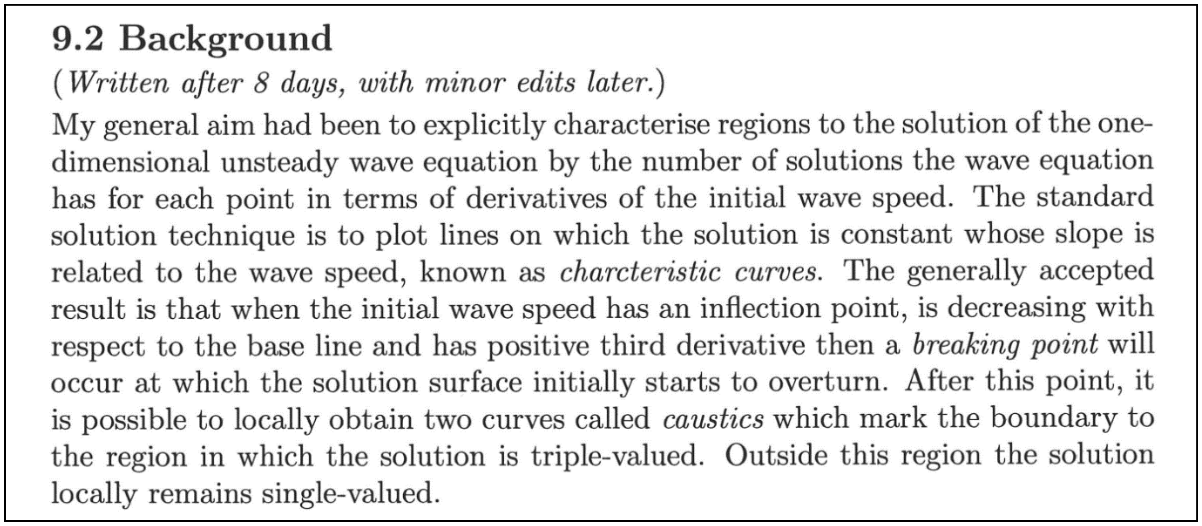
I WOULD LIKE TO VIEW THIS ENTIRE PROCESS AS A CASE STUDY OF MATHEMATICAL KNOWLEDGE (RE)ACQUISITION, REPRENTATION & INFERENCE.

LEVEL 3. └ Compare with MUSCADET, Michener etc.. (Mandl & Levin).

Figure 5. Second example of plan writing.

5.2 Activity Transcripts

A mathematical activity transcript is a detailed account of a specific mathematical experience. It combines notes written at the time of the activity with an account of what the mathematician was thinking when he/she created these notes. It may also include other forms of writing, such as an introduction to the context of the experience and a reflection on the experience. Figures 6a to 6d provide four extracts from an activity transcript relating to non-linear wave theory: an introduction, written 8 days after the activity; notes written during the activity; an account of the activity, also written 8 days after it occurred; and a review or reflection, written about 3 weeks later. The whole activity transcript is provided in Appendix B. Figures 6b and 6c include a mistake which was discovered only during the reflection, in Figure 6d. This has been included to illustrate how actual mathematical activity sometimes contains mistakes which may be corrected at a later stage. Due to the multiple nature of its content, an activity transcript relates to the incubation, illumination and verification stages in the mathematical creative process. It is a dynamic technique, as the critical thinking of the mathematician is seen to change through the transcript. In essence, it captures the process of creating mathematics.

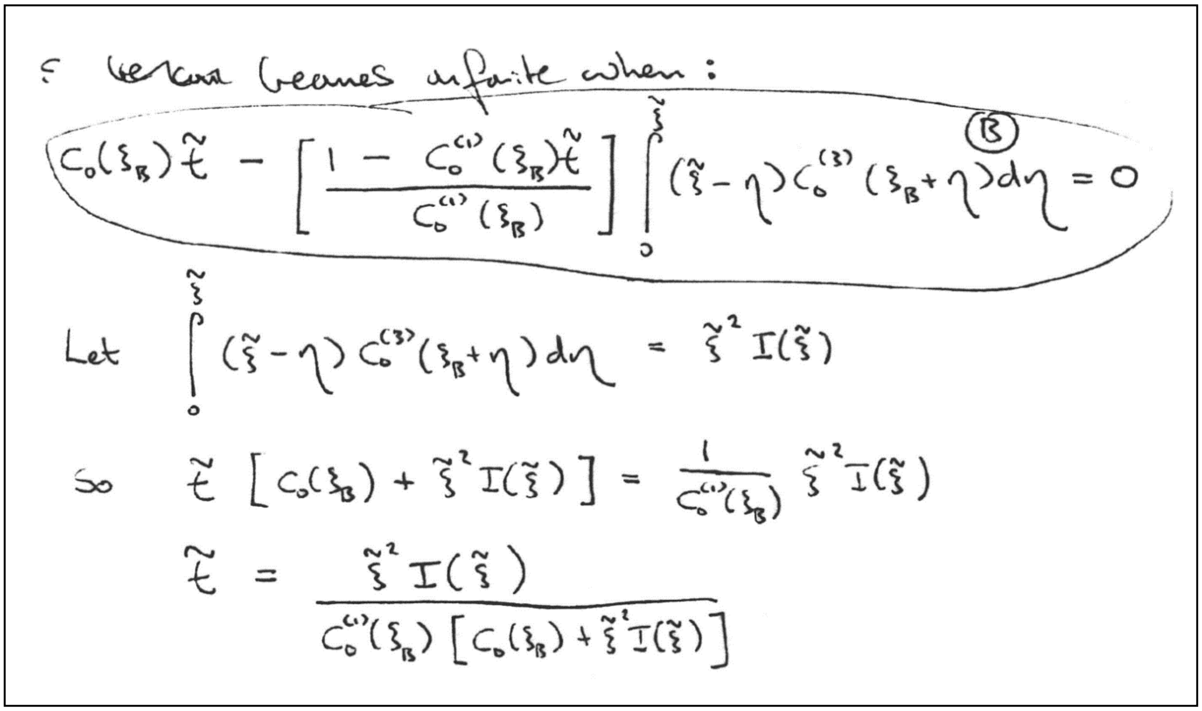


9.2 Background

(*Written after 8 days, with minor edits later.*)

My general aim had been to explicitly characterise regions to the solution of the one-dimensional unsteady wave equation by the number of solutions the wave equation has for each point in terms of derivatives of the initial wave speed. The standard solution technique is to plot lines on which the solution is constant whose slope is related to the wave speed, known as *charcteristic curves*. The generally accepted result is that when the initial wave speed has an inflection point, is decreasing with respect to the base line and has positive third derivative then a *breaking point* will occur at which the solution surface initially starts to overturn. After this point, it is possible to locally obtain two curves called *caustics* which mark the boundary to the region in which the solution is triple-valued. Outside this region the solution locally remains single-valued.

Figure 6a. Background statement relating to the example of mathematical activity.



∴ [illegible] becomes infinite when :

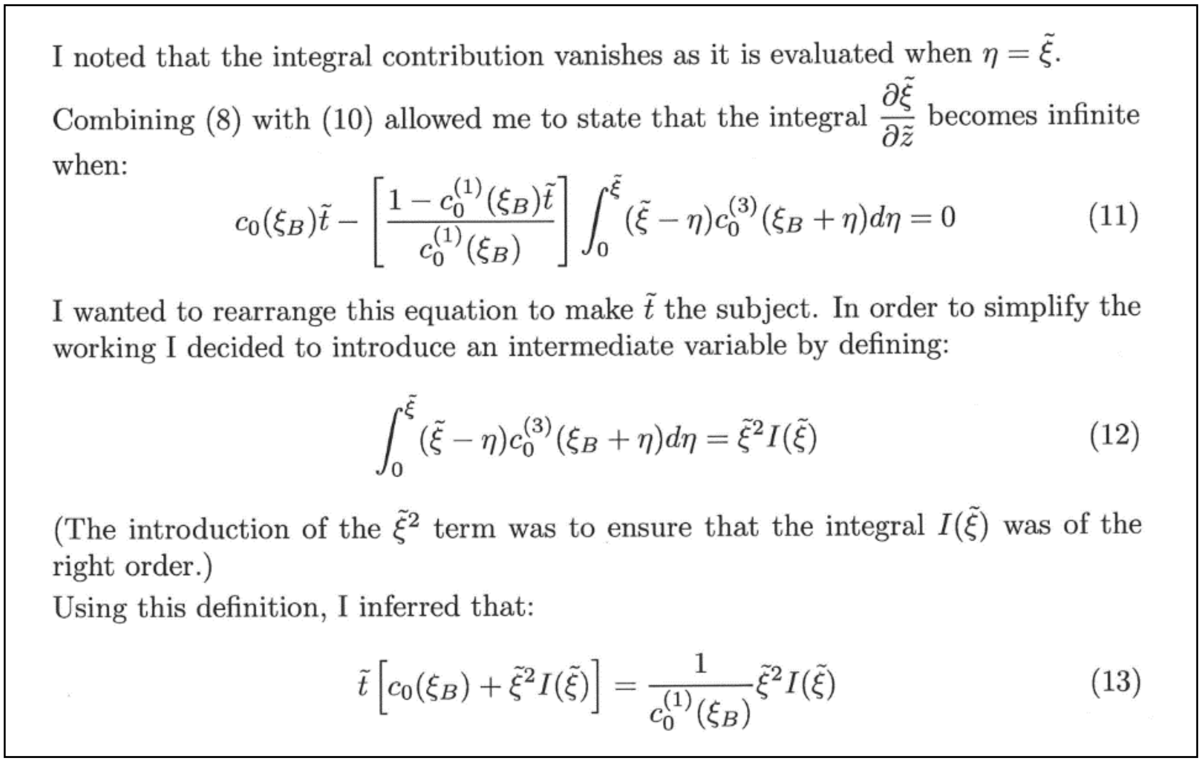
$$C_0(\xi_B)\tilde{t} - \left[\frac{1 - C_0^{(1)}(\xi_B)\tilde{t}}{C_0^{(1)}(\xi_B)}\right]\int_0^{\tilde{\xi}}(\tilde{\xi}-\eta)C_0^{(3)}(\xi_B+\eta)\,d\eta = 0 \quad Ⓑ$$

Let $\displaystyle\int_0^{\tilde{\xi}}(\tilde{\xi}-\eta)C_0^{(3)}(\xi_B+\eta)\,d\eta = \tilde{\xi}^2 I(\tilde{\xi})$

So $\tilde{t}\left[C_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right] = \dfrac{1}{C_0^{(1)}(\xi_B)}\tilde{\xi}^2 I(\tilde{\xi})$

$$\tilde{t} = \frac{\tilde{\xi}^2 I(\tilde{\xi})}{C_0^{(1)}(\xi_B)\left[C_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right]}$$

Figure 6b. Extract from the mathematical transcript.



I noted that the integral contribution vanishes as it is evaluated when $\eta = \tilde{\xi}$.

Combining (8) with (10) allowed me to state that the integral $\dfrac{\partial\tilde{\xi}}{\partial\tilde{z}}$ becomes infinite when:

$$c_0(\xi_B)\tilde{t} - \left[\frac{1-c_0^{(1)}(\xi_B)\tilde{t}}{c_0^{(1)}(\xi_B)}\right]\int_0^{\tilde{\xi}}(\tilde{\xi}-\eta)c_0^{(3)}(\xi_B+\eta)d\eta = 0 \tag{11}$$

I wanted to rearrange this equation to make $\tilde{t}$ the subject. In order to simplify the working I decided to introduce an intermediate variable by defining:

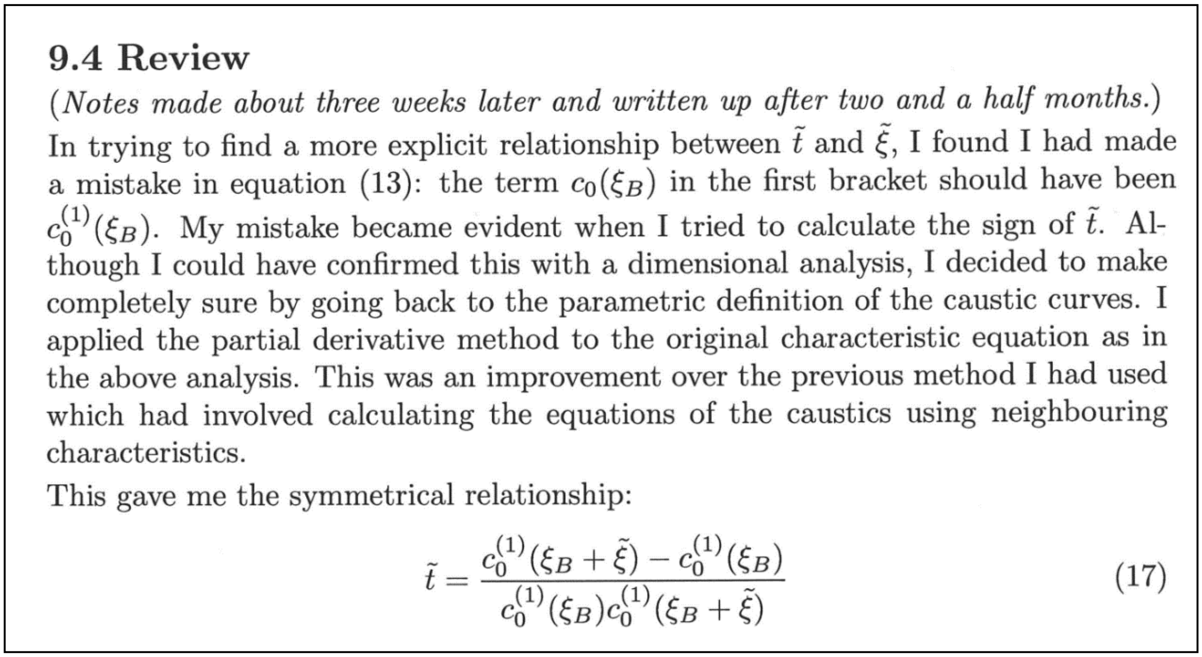
$$\int_0^{\tilde{\xi}}(\tilde{\xi}-\eta)c_0^{(3)}(\xi_B+\eta)d\eta = \tilde{\xi}^2 I(\tilde{\xi}) \tag{12}$$

(The introduction of the $\tilde{\xi}^2$ term was to ensure that the integral $I(\tilde{\xi})$ was of the right order.)

Using this definition, I inferred that:

$$\tilde{t}\left[c_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right] = \frac{1}{c_0^{(1)}(\xi_B)}\tilde{\xi}^2 I(\tilde{\xi}) \tag{13}$$

Figure 6c. Narrative for the mathematical transcript.



9.4 Review

(*Notes made about three weeks later and written up after two and a half months.*)

In trying to find a more explicit relationship between $\tilde{t}$ and $\tilde{\xi}$, I found I had made a mistake in equation (13): the term $c_0(\xi_B)$ in the first bracket should have been $c_0^{(1)}(\xi_B)$. My mistake became evident when I tried to calculate the sign of $\tilde{t}$. Although I could have confirmed this with a dimensional analysis, I decided to make completely sure by going back to the parametric definition of the caustic curves. I applied the partial derivative method to the original characteristic equation as in the above analysis. This was an improvement over the previous method I had used which had involved calculating the equations of the caustics using neighbouring characteristics.

This gave me the symmetrical relationship:

$$\tilde{t} = \frac{c_0^{(1)}(\xi_B+\tilde{\xi}) - c_0^{(1)}(\xi_B)}{c_0^{(1)}(\xi_B)c_0^{(1)}(\xi_B+\tilde{\xi})} \tag{17}$$

Figure 6d. Reflection on the mathematical activity.

Compared with Tall's (1980) account of his discovery of a new mathematical concept, activity transcripts are more detailed and more integrated as a single document describing a single event. Consistent with Figure 6d, he recounted making many small errors during his discovery process. Regarding the danger of post-rationalisation, he stated, "I am very suspicious of mathematicians who recall how they did research *without taking careful notes at the time*" (p. 24). The detailed original notes form the basis of activity transcripts, increase the accuracy of the post rationalisations made in the accounts of the experiences and reduce the applicability of Nisbett and Wilson's (1977) criticisms of the accuracy of verbal reports on mental processes.

Craig (2011) recently used journals of problem solving activities with first year mathematics undergraduates. The students were asked to write explanatory paragraphs of their problem solving behaviour. These were analysed using Waywood's (1992) classification of student mathematical journal entries: recounting—reporting what happened, summarising—codifying and organising content, and dialogue—showing an interaction between ideas. Craig found a strong correlation between the journal entries and Waywood's classification scheme. She also deliberately included an example containing a mistake. The approach taken in Figures 6a to 6d are a combination of recounting (in the transcript notes themselves and the account) and dialogue (in the reflection).

In the wider scientific context, a famous example of an activity transcript is Faraday's diary (1932-1936), containing transcripts of his original notes whilst retaining his original illustrations. Parts of these have been analysed by researchers. For example, Gooding (1990) devised a formal language for investigating the creative process by which Faraday discovered the electric motor. However, the scientific discovery process is slightly different from the mathematical one as it generally requires constructing apparatus and carrying out experiments in order to test hypotheses. Furthermore, West (1992) asserted that Faraday's particular approach may be attributable to his being dyslexic and thus not generalizable to an understanding of the nature of scientific creativity.

5.3 Concept Maps

According to Novak and Cañas (2008), concept maps are

> graphical tools for organizing and representing knowledge. They include concepts, usually enclosed in circles or boxes of some type, and relationships between concepts indicated by a connecting line linking two concepts. Words on the line, referred to as linking words or linking phrases, specify the relationship between the two concepts. (p. 1)

However, according to Gaines and Shaw (1995), the term *concept map* is used to "encompass a wide range of diagrammatic knowledge representations" (p. 334); they went on to provide a more formal definition of a concept map which is beyond the scope of this paper. In any case, the practice of using concept maps is often different from formal attempts to define what they are.

In addition to Novak and Cañas' (2008) statement above, the linking lines between concepts are sometimes directed using arrows. Groups of concepts are sometimes identified by drawing a shape around them, such as a rectangle, and also labelled. The naming of a link between two concepts can be interpreted formally as a predicated proposition of the form *LinkName(Concept1, Concept2)*. The physical proximity of concepts can also be seen as implying an association between two concepts (Simone et al., 2001).

Concept maps are easy to create but are often dismissed by academics with a "traditional dualistic orientation" (Hung, D., Looi, C.-K., & Koh, T.-S., 2004, p. 193) as lacking objective interpretation. However, as Gaines and Shaw (1995) observed, all knowledge is subject to interpretation by a reference community, and "there is an exact parallel between natural language and visual language—the abstract grammatical structure and their expressions in a medium take on meaning only through the practices of a community of discourse" (p. 335). However, this is disputed by Hoey (2005), who claimed that corpora are "central to a proper understanding of discourses as a whole" (p. 150). The subject of corpora is revisited in Section 6 below.

Whilst concept maps are used for different purposes, the purpose relevant to this paper is the visual representation and communication of tacit knowledge from experts about their domains of expertise. Examples of concept maps from my PhD thesis (Samuels, 2000) are provided in Appendix C. An example is not provided in the main paper, as they do not relate to the same piece of mathematics as the other three examples of the techniques presented in this section. They differ in degree of structure and breadth of knowledge content. All these maps were created for my own benefit to aid the representation and communication of mathematical knowledge. They can be created at the preparation and incubation stage of mathematical creativity because reflection on conceptual relationships could be seen as a precursor to a new mathematical discovery, such as Kaufman's (2012) anthropological presentation of the discovery of a new duality transform. Generally, a concept map is a static data capturing technique. It can also be used in the planning, translating and reviewing sub-processes of the composition process (see Figure 11 below).

Concept maps are common in secondary education, especially in science (Novak & Cañas, 2008). Bolte (1999) suggested they could be used as a complementary assessment technique in undergraduate mathematics. More recently, Lavigne et al. (2008) used them as a research tool to investigate students' mental representations of inferential statistics. Mac Lane (1986) used concept maps to describe the interconnection between concepts in different areas of mathematics. Otherwise, the use of concept maps by research mathematicians is rare. Concept maps also relate to the writing process, especially pre-writing (Ojima, 2006).

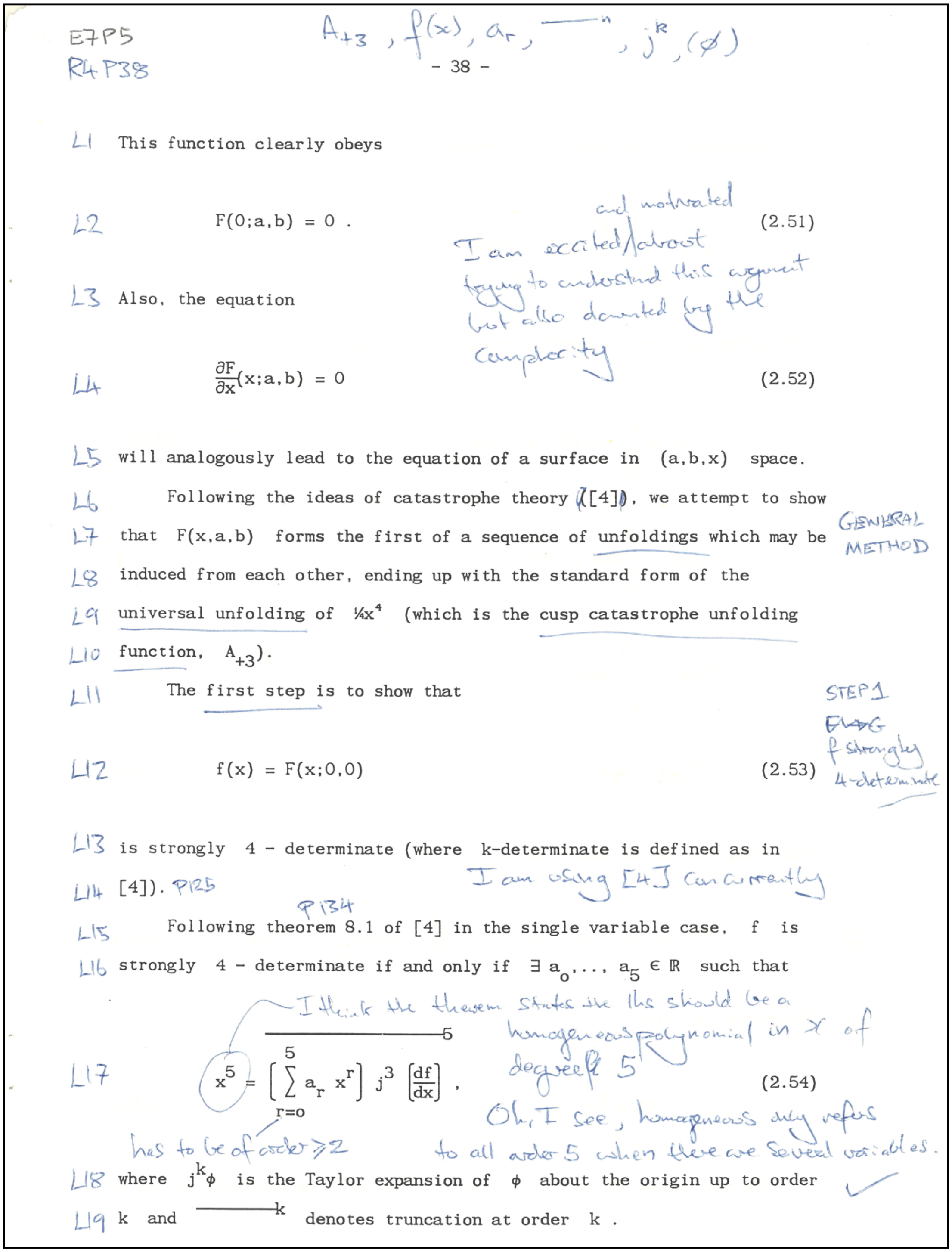
The *Structure of the Observed Learning Outcome* (SOLO) taxonomy provides a knowledge representation similar to concept maps, known as response structures. Within this taxonomy, concepts are labelled in different types: data or cues, concepts or processes, abstract concepts or abstract processes, and responses. The structures created are more dynamic and represent the way an individual's conceptual understanding develops over time. Chick (1998) applied the SOLO taxonomy to her doctoral research in abstract algebra. However, concept maps are promoted here because they are perceived as being more practical for research mathematicians to understand and use.

5.4 Annotated Drafts and Transcripts

The final data capturing technique introduced in this paper is an annotated draft and transcript. The idea for this technique was derived from the version of T.S. Eliot's (1971) poem *The Waste Land* edited by his first wife, who made facsimile copies of the pages of the original draft, numbered the lines and then transcribed both the draft and the different annotations on the opposite page. My approach is based on annotations I made when re-reading extracts of my own internal reports. I have numbered the lines and transcribed all the comments but not the original text (as this was already typed). Each page of the extract begins with a list of the variables introduced thereon in order to provide a measure of the working memory load required by the reader.

An example page of an extract is given in Figure 7 with its transcript given in Figure 8 (note the emotional reflection written next to Lines 1 to 4 and the "seeing" in the comment next to

Line 17). The whole of this extract and its transcript are provided in Appendix D (note: "Report 4" to which this extract refers is (Samuels, 1989)). Whilst annotating drafts is not a new idea, their use in capturing critical thinking in the composition of advanced mathematics is believed to be new. As with Eliot's (1971) facsimile and transcript edition of his draft, of particular relevance is the social context in which the drafts are created.



E7P5
R4P38

A_{+3}, $f(x)$, a_r, $\overline{\quad\quad}^{\,n}$, j^k, (ϕ)

- 38 -

L1 This function clearly obeys

L2 $$F(0;a,b) = 0 \,. \tag{2.51}$$

I am excited and motivated about trying to understand this argument but also daunted by the complexity

L3 Also, the equation

L4 $$\frac{\partial F}{\partial x}(x;a,b) = 0 \tag{2.52}$$

L5 will analogously lead to the equation of a surface in (a,b,x) space.

L6 Following the ideas of catastrophe theory ([4]), we attempt to show
L7 that F(x,a,b) forms the first of a sequence of unfoldings which may be
L8 induced from each other, ending up with the standard form of the
L9 universal unfolding of $\frac{1}{4}x^4$ (which is the cusp catastrophe unfolding
L10 function, A_{+3}).

GENERAL METHOD

L11 The first step is to show that

STEP 1
~~FLAG~~
f strongly 4-determinate

L12 $$f(x) = F(x;0,0) \tag{2.53}$$

L13 is strongly 4 - determinate (where k-determinate is defined as in
L14 [4]). P125

I am using [4] concurrently

P134

L15 Following theorem 8.1 of [4] in the single variable case, f is
L16 strongly 4 - determinate if and only if $\exists\, a_o, \ldots, a_5 \in \mathbb{R}$ such that

I think the theorem states the lhs should be a homogeneous polynomial in x of degree 5

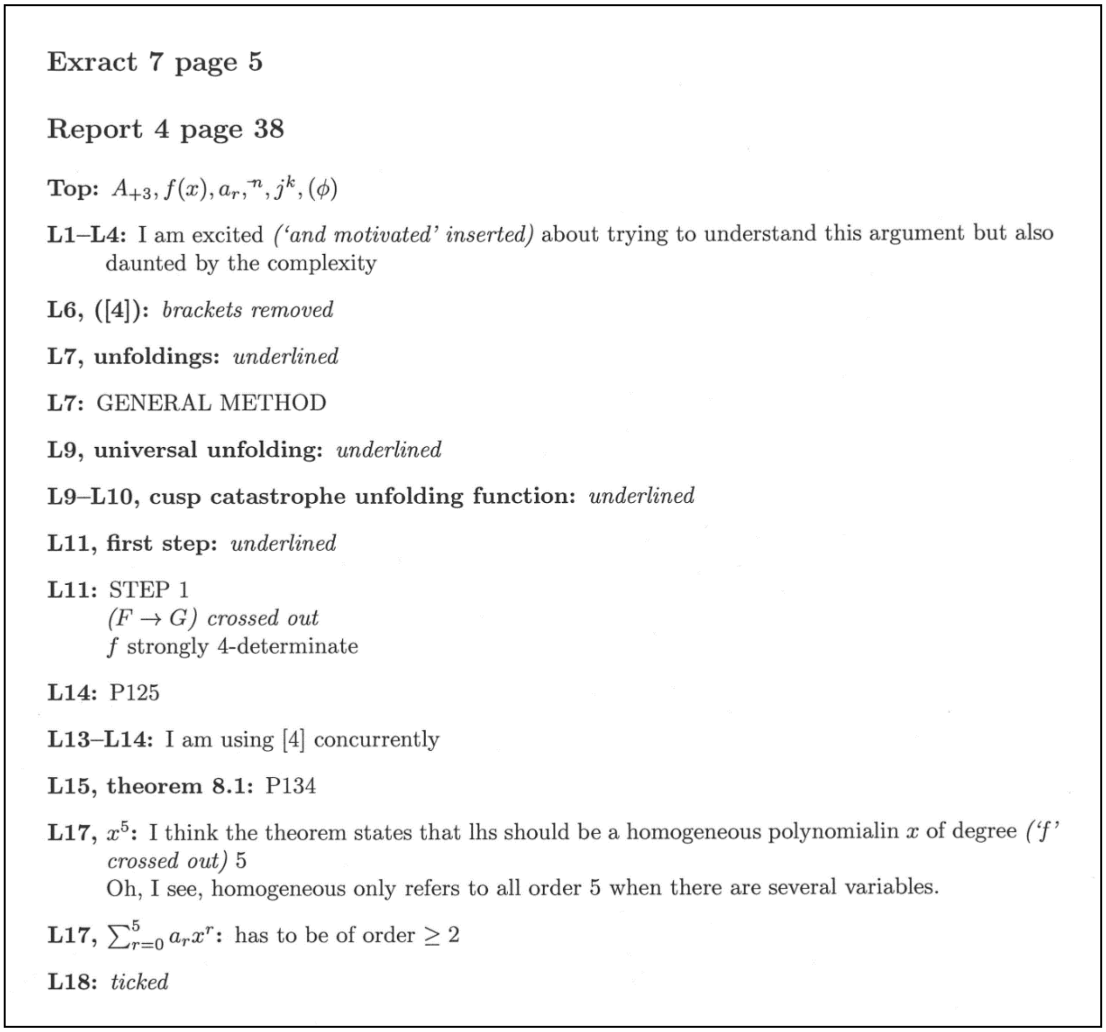
L17 $$x^5 = \overline{\left[\sum_{r=o}^{5} a_r x^r\right] j^3 \left[\frac{df}{dx}\right]}^{\,5} \,, \tag{2.54}$$

has to be of order ≥2

Oh, I see, homogeneous only refers to all order 5 when there are several variables.

L18 where $j^k\phi$ is the Taylor expansion of ϕ about the origin up to order ✓
L19 k and $\overline{\quad\quad}^{\,k}$ denotes truncation at order k .

Figure 7. Example annotated draft.



Exract 7 page 5

Report 4 page 38

Top: $A_{+3}, f(x), a_r, {}^{-n}, j^k, (\phi)$

L1–L4: I am excited *('and motivated' inserted)* about trying to understand this argument but also daunted by the complexity

L6, ([4]): *brackets removed*

L7, unfoldings: *underlined*

L7: GENERAL METHOD

L9, universal unfolding: *underlined*

L9–L10, cusp catastrophe unfolding function: *underlined*

L11, first step: *underlined*

L11: STEP 1
$(F \to G)$ crossed out
f strongly 4-determinate

L14: P125

L13–L14: I am using [4] concurrently

L15, theorem 8.1: P134

L17, x^5: I think the theorem states that lhs should be a homogeneous polynomialin x of degree *('f' crossed out)* 5
Oh, I see, homogeneous only refers to all order 5 when there are several variables.

L17, $\sum_{r=0}^{5} a_r x^r$: has to be of order ≥ 2

L18: *ticked*

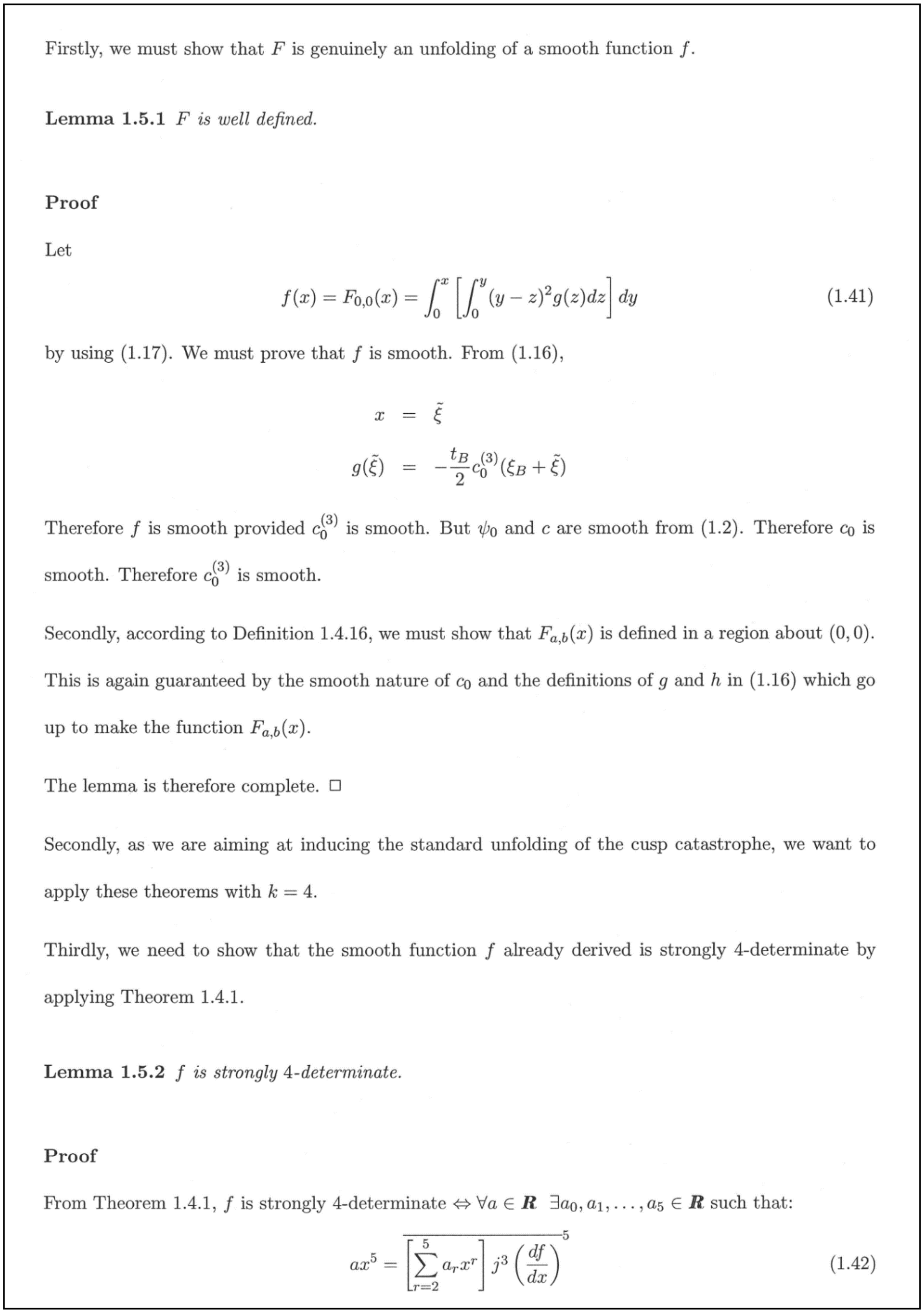
Figure 8. Example transcript of annotated draft.

The annotated draft and transcript technique is dynamic and clearly fits in with the writing sub-process of reviewing. However, it could also be appropriate for the preparation stage in the mathematical creativity process if the draft text needs to be improved substantially. This was certainly the case with my reflections on my internal reports. Part of the final proof relating to the extract provided in Figures 7 and 8 is given in Figure 9. The whole of the deductive form of the proof is provided in Appendix E. The content of the final version of the proof looks very different from that in the internal report.

Whilst the publication of results within internal departmental reports may not be so common, it is usual for mathematical ideas and results to be communicated first in an informal or semi-formal setting before they are submitted to and published in journal articles. Therefore, an annotated draft and transcript approach may be widely applicable to mathematical creativity and writing.

6. Discussion

The purpose of this paper has been to present four practical techniques which enable mathematicians to capture and communicate their critical thinking processes when creating and composing advanced mathematical knowledge. The use of these techniques requires a shift in



Firstly, we must show that F is genuinely an unfolding of a smooth function f.

Lemma 1.5.1 *F is well defined.*

Proof

Let

$$f(x) = F_{0,0}(x) = \int_0^x \left[\int_0^y (y-z)^2 g(z) dz \right] dy \tag{1.41}$$

by using (1.17). We must prove that f is smooth. From (1.16),

$$\begin{aligned} x &= \tilde{\xi} \\ g(\tilde{\xi}) &= -\frac{t_B}{2} c_0^{(3)} (\xi_B + \tilde{\xi}) \end{aligned}$$

Therefore f is smooth provided $c_0^{(3)}$ is smooth. But ψ_0 and c are smooth from (1.2). Therefore c_0 is smooth. Therefore $c_0^{(3)}$ is smooth.

Secondly, according to Definition 1.4.16, we must show that $F_{a,b}(x)$ is defined in a region about $(0,0)$. This is again guaranteed by the smooth nature of c_0 and the definitions of g and h in (1.16) which go up to make the function $F_{a,b}(x)$.

The lemma is therefore complete. □

Secondly, as we are aiming at inducing the standard unfolding of the cusp catastrophe, we want to apply these theorems with $k = 4$.

Thirdly, we need to show that the smooth function f already derived is strongly 4-determinate by applying Theorem 1.4.1.

Lemma 1.5.2 *f is strongly 4-determinate.*

Proof

From Theorem 1.4.1, f is strongly 4-determinate $\Leftrightarrow \forall a \in \boldsymbol{R} \;\; \exists a_0, a_1, \ldots, a_5 \in \boldsymbol{R}$ such that:

$$ax^5 = \overline{\left[\sum_{r=2}^{5} a_r x^r \right] j^3 \left(\frac{df}{dx} \right)}^{5} \tag{1.42}$$

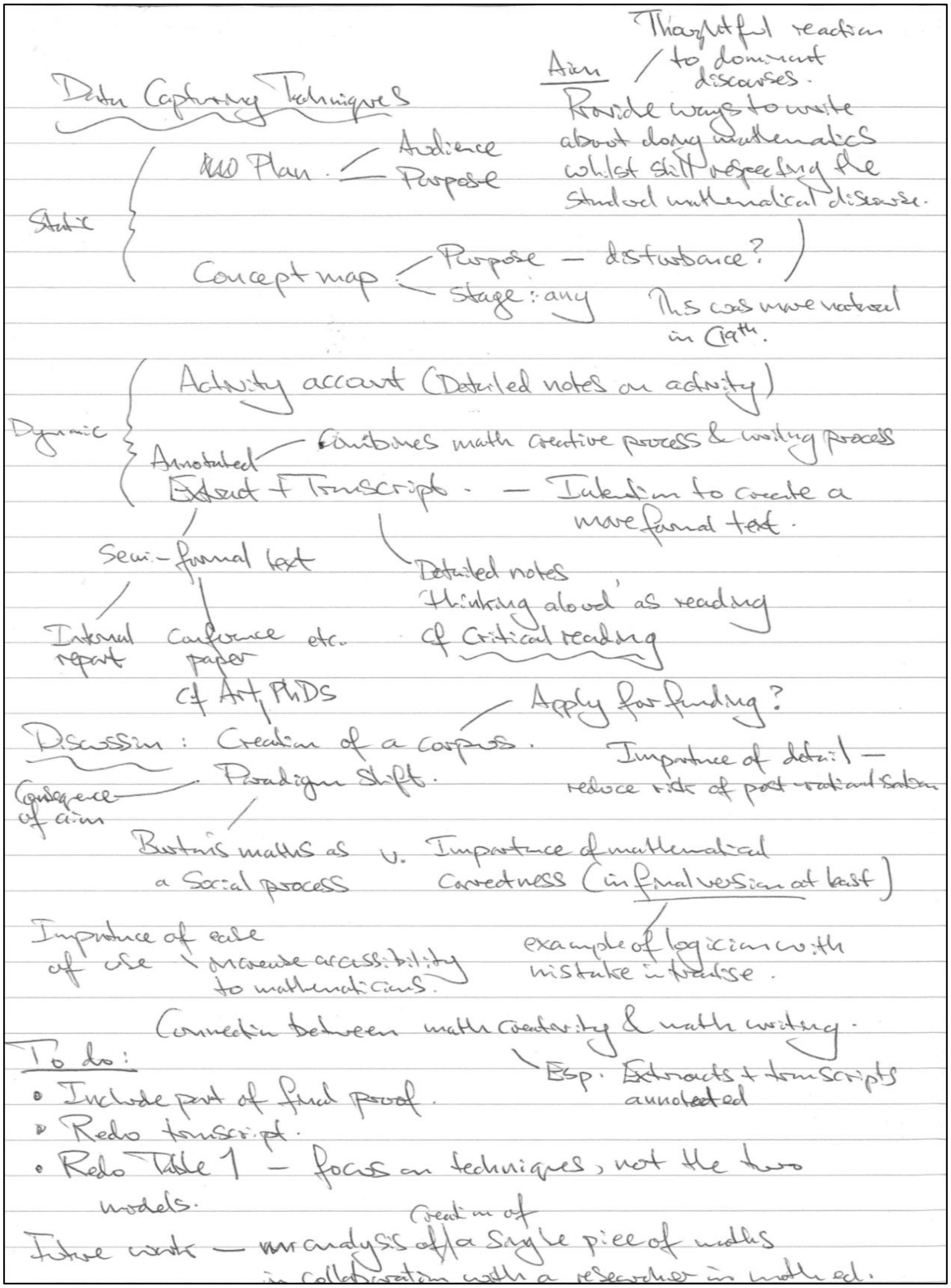
Figure 9. Extract from final published version of proof.

perception of the role of mathematicians from research subject or co-researcher in research initiated by a behavioural researcher to transcript provider. Furthermore, their use is not in opposition to traditional mathematical creative activity and the standard, product-orientated mathematical writing genre but rather they can work alongside them, enabling mathematicians to express their thinking processes and recapture the narrative writing style that was common in a previous age (Solomon & O'Neill, 1998).

All four of these techniques are relatively easy to use, making them practical and accessible to mathematicians. As the information is coming directly from the mathematicians and relates to their actual creative and writing processes, these techniques are more appropriate and have a greater potential to provide accurate data on critical thinking than the traditional data capturing techniques used by behavioural researchers outlined in Section 3. The two dynamic techniques, activity transcripts and annotated drafts and transcripts, emphasise the importance of capturing detail, potentially leading to accurate post-rationalisations. In particular, activity transcripts are promoted because they have the potential to capture detailed thought processes during the mathematical creative process.

This paper has explored the nature of critical thinking in an advanced mathematical context. Critical thinking in mathematics is fundamentally good mathematical thinking, which primarily is being able to create and identify mathematically correct arguments. Whilst it has not been the purpose of this paper to analyse the critical thinking within the examples of the proposed techniques, the correction of a mistake in Figures 6b, 6c and 6d illustrates it. The examples provided also illustrate some of the other forms of critical thinking in mathematics discussed in Section 3, such as deciding what to do next when creating mathematics, "seeing" results intuitively and planning both mathematical activity and mathematical writing.

A theme common to the examples of these techniques provided is the importance of the social context in which they have been created. Therefore, in order to encourage other mathematicians to engage socially with these techniques, creating a corpus of advanced mathematical process data which mathematical behavioural researchers can study is proposed. The figures in this paper and the supplementary data supplied in the appendices are my initial contribution to such a corpus. Such an approach would be similar to that taken in the Digital Variants corpus (Björk & Holmquist, 1998) (http://www.digitalvariants.org/) which enables living authors to present texts created at different stages of the writing process. Wolska et al. (2004) created a corpus of tutorial dialogs of people with different levels of mathematical ability proving theorems in basic set theory. The data provided with this paper, especially the process of creating a deductive proof applying catastrophe theory to nonlinear wave theory, could form a joint research study with mathematical behavioural researchers. Finally, at the meta level, Figure 10 below is a hybrid of a writing plan and a concept map I produced during the process of creating this paper.



Aim / Thoughtful reaction to dominant discourses.

Data Capturing Techniques

Provide ways to write about doing mathematics whilst still respecting the standard mathematical discourse.

Static

AWO Plan. Audience
Purpose

Concept map: Purpose — disturbance?
Stage: any

This was more natural in C19th.

Dynamic

Activity account (Detailed notes on activity)

Annotated Extract + Transcript. — Intention to create a more formal text.

Combines math creative process & writing process

Semi-formal text

Internal report

Conference etc. paper

Detailed notes
'Thinking aloud' as reading
Cf Critical reading

Cf Art PhDs

Apply for funding?

Discussion: Creation of a corpus.

Importance of detail — reduce risk of post-rationalisation

Consequence of aim — Paradigm shift.

Burton's maths as a social process v. Importance of mathematical correctness (in final version at least)

example of logician with mistake in treatise.

Importance of ease of use \ increase accessibility to mathematicians.

Connection between math creativity & math writing.

Esp. Extracts + transcripts annotated

To do:

- Include part of final proof.
- Redo transcript.
- Redo Table 1 — focus on techniques, not the two models.

Future work — ~~an~~ analysis of Creation of a single piece of maths in collaboration with a researcher in math ed.

Figure 10. Hybrid writing plan/concept of this paper.

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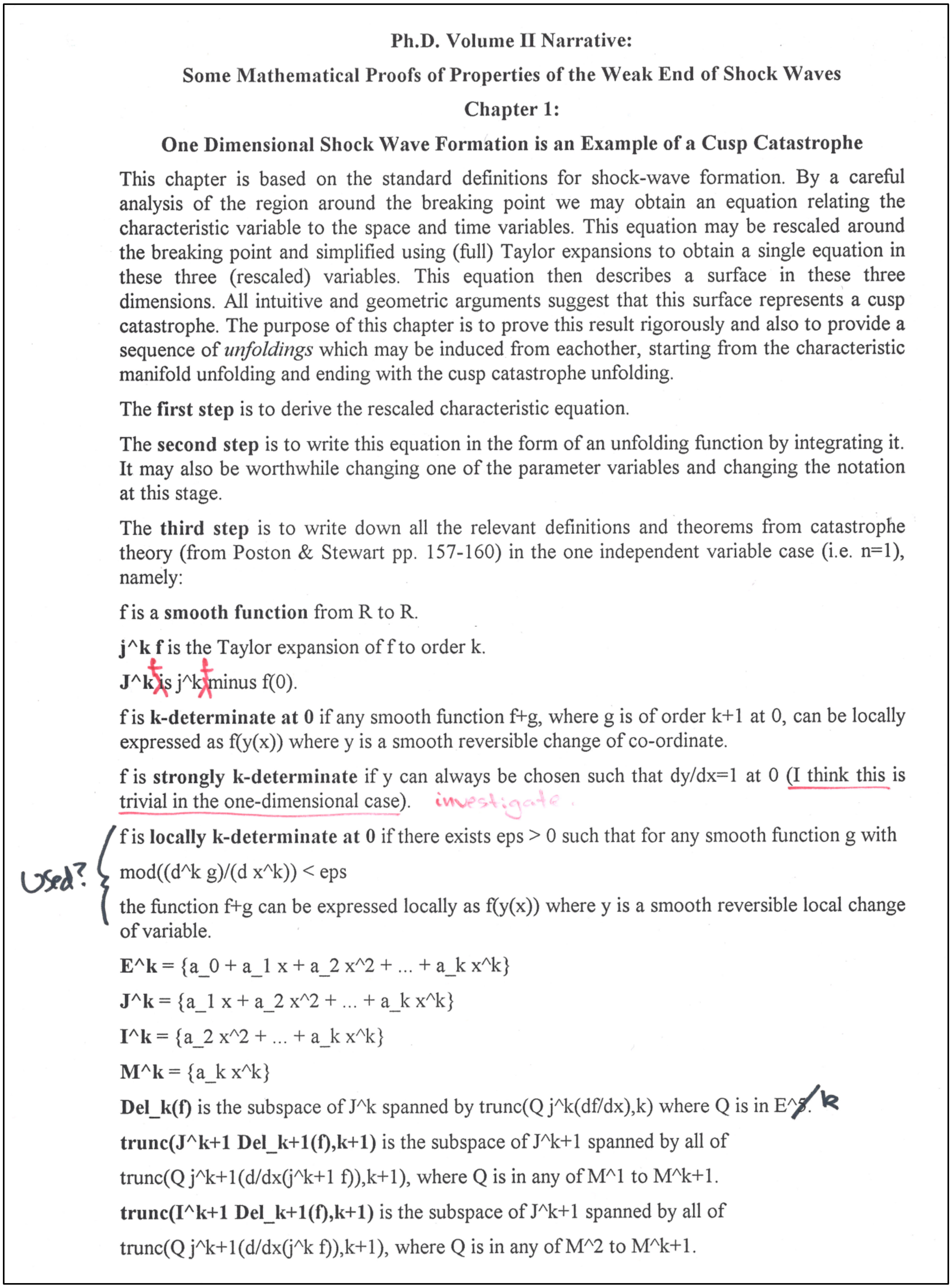
Appendix A

Other Pages of First Example of a Mathematical Proof Plan



Ph.D. Volume II Narrative:

Some Mathematical Proofs of Properties of the Weak End of Shock Waves

Chapter 1:

One Dimensional Shock Wave Formation is an Example of a Cusp Catastrophe

This chapter is based on the standard definitions for shock-wave formation. By a careful analysis of the region around the breaking point we may obtain an equation relating the characteristic variable to the space and time variables. This equation may be rescaled around the breaking point and simplified using (full) Taylor expansions to obtain a single equation in these three (rescaled) variables. This equation then describes a surface in these three dimensions. All intuitive and geometric arguments suggest that this surface represents a cusp catastrophe. The purpose of this chapter is to prove this result rigorously and also to provide a sequence of *unfoldings* which may be induced from eachother, starting from the characteristic manifold unfolding and ending with the cusp catastrophe unfolding.

The **first step** is to derive the rescaled characteristic equation.

The **second step** is to write this equation in the form of an unfolding function by integrating it. It may also be worthwhile changing one of the parameter variables and changing the notation at this stage.

The **third step** is to write down all the relevant definitions and theorems from catastrophe theory (from Poston & Stewart pp. 157-160) in the one independent variable case (i.e. n=1), namely:

f is a **smooth function** from R to R.

j^k f is the Taylor expansion of f to order k.

J^k f is j^k f minus f(0).

f is **k-determinate at 0** if any smooth function f+g, where g is of order k+1 at 0, can be locally expressed as f(y(x)) where y is a smooth reversible change of co-ordinate.

f is **strongly k-determinate** if y can always be chosen such that dy/dx=1 at 0 (I think this is trivial in the one-dimensional case). investigate .

Used?

f is **locally k-determinate at 0** if there exists eps > 0 such that for any smooth function g with

mod((d^k g)/(d x^k)) < eps

the function f+g can be expressed locally as f(y(x)) where y is a smooth reversible local change of variable.

E^k = {a_0 + a_1 x + a_2 x^2 + ... + a_k x^k}

J^k = {a_1 x + a_2 x^2 + ... + a_k x^k}

I^k = {a_2 x^2 + ... + a_k x^k}

M^k = {a_k x^k}

Del_k(f) is the subspace of J^k spanned by trunc(Q j^k(df/dx),k) where Q is in E^~~5~~. k

trunc(J^k+1 Del_k+1(f),k+1) is the subspace of J^k+1 spanned by all of

trunc(Q j^k+1(d/dx(j^k+1 f)),k+1), where Q is in any of M^1 to M^k+1.

trunc(I^k+1 Del_k+1(f),k+1) is the subspace of J^k+1 spanned by all of

trunc(Q j^k+1(d/dx(j^k f)),k+1), where Q is in any of M^2 to M^k+1.

The **codimension of f at 0, cod(f)** is the codimension of Del_k(f) in J^k for any k for which f is k-determinate.

An **r-unfolding of f at 0** is a function F:R^(r+1) -> R,
(x,t_1,..,t_r) -> F_t(x)

such that F_0(x) = f(x), defined in a region around (0,...,0).

A d-unfolding G is **induced** from F by three mappings, defined in a region about the origin:

e:R^d -> R^r, (s_1,...,s_d) -> (e_1(s),...,e_r(s))
y:R^(d+1) -> R (x,s) -> y(x,s)
gam: R^d -> R

provided

G(x,s) = F(y_s(x),e(s)) + gam(s)

don't have to worry about γ in our case by integral const" of F.

Two r-unfoldings are **strongly equivalent** to eachother if they can be induced from eachother with (dpar e_i)/(dpar t_j) = delta_ij at 0.

An r-unfolding of f at 0 is **versal** if all other unfoldings of f at 0 can be induced from it.

An r-unfolding of f at 0 is **universal** if it is versal and r = cod(f).

If F is an unfolding of f, set

v^k_1(F) = dpar/(dpar t) (J^k(F_(t_1,0,...,0)))
v^k_2(F) = dpar/(dpar t) (J^k(F_(0,t_2,0,...,0)))
...
v^k_r(F) = dpar/(dpar t) (J^k(F_(0,...,0,t_r)))

Then **V^k(F)** = span{v^k_1(F),...,v^k_r(F)}

Theorem 8.4

f is k-determinate if and only if, for all P in M^(k+1),

M^(k+1) subspace trunc(J^(k+1) Del_(k+1) (j^k f + P), k+1).

Theorem 8.6

An r-unfolding F of f, where f is k-determinate, is versal if and only if V^k(F) and Del_k(f) are transverse subspaces of J^k.

Corollary

If f is k-determinate, then a universal unfolding for f may be constructed by choosing a cobasis v_1,...,v_c for Del_k(f) in J^k and setting

F(x,t_1,...,t_c) = f(x) + t_1 v_1(x) + ... + t_c v_c(x)

Theorem 8.7

A versal unfolding F of f is strongly equivalent to the truncated unfolding

j^p f(x) + t_1 J^q(dpar/(dpar t)_1 F_(t_1,0,...,0)) + ... + t_r J^q(dpar(dpar t)_r F_(0,...,0,t_r))

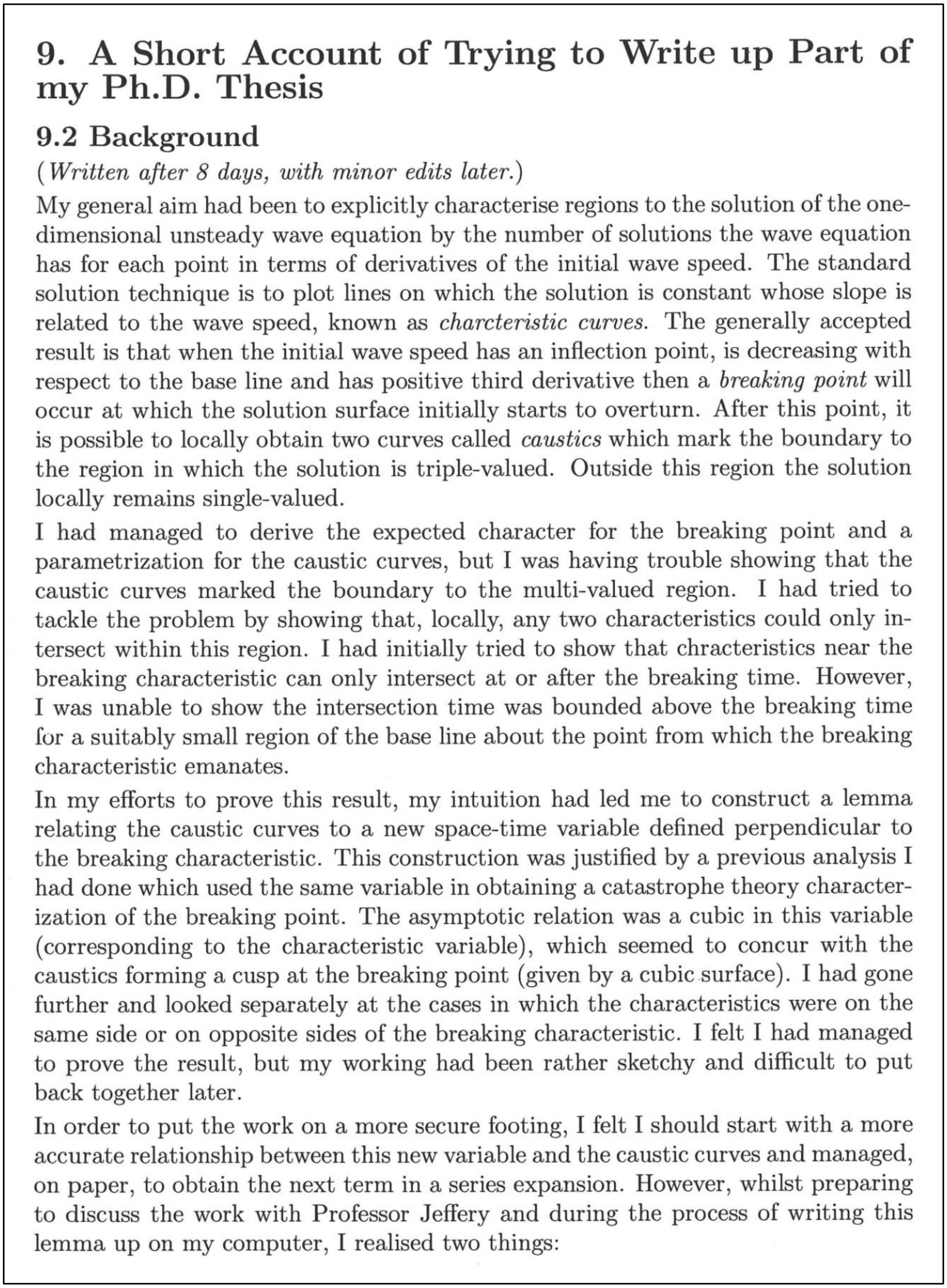
if f is strongly k-determinate, k >= 3, and

p >= 2k-3, q >= k-2 when M^(k-1) subspace Del_(k+1) (f)
p >= 2k-2, q >= k-1 when M^k subspace Del_(k+1) (f)
p >= 2k-1, q >= k when M^(k+1) subspace Del_(k+1) (f)

At least one of these cases must hold.

Appendix B

Whole Example Activity Transcript



9. A Short Account of Trying to Write up Part of my Ph.D. Thesis

9.2 Background

(*Written after 8 days, with minor edits later.*)

My general aim had been to explicitly characterise regions to the solution of the one-dimensional unsteady wave equation by the number of solutions the wave equation has for each point in terms of derivatives of the initial wave speed. The standard solution technique is to plot lines on which the solution is constant whose slope is related to the wave speed, known as *charcteristic curves*. The generally accepted result is that when the initial wave speed has an inflection point, is decreasing with respect to the base line and has positive third derivative then a *breaking point* will occur at which the solution surface initially starts to overturn. After this point, it is possible to locally obtain two curves called *caustics* which mark the boundary to the region in which the solution is triple-valued. Outside this region the solution locally remains single-valued.

I had managed to derive the expected character for the breaking point and a parametrization for the caustic curves, but I was having trouble showing that the caustic curves marked the boundary to the multi-valued region. I had tried to tackle the problem by showing that, locally, any two characteristics could only intersect within this region. I had initially tried to show that chracteristics near the breaking characteristic can only intersect at or after the breaking time. However, I was unable to show the intersection time was bounded above the breaking time for a suitably small region of the base line about the point from which the breaking characteristic emanates.

In my efforts to prove this result, my intuition had led me to construct a lemma relating the caustic curves to a new space-time variable defined perpendicular to the breaking characteristic. This construction was justified by a previous analysis I had done which used the same variable in obtaining a catastrophe theory characterization of the breaking point. The asymptotic relation was a cubic in this variable (corresponding to the characteristic variable), which seemed to concur with the caustics forming a cusp at the breaking point (given by a cubic surface). I had gone further and looked separately at the cases in which the characteristics were on the same side or on opposite sides of the breaking characteristic. I felt I had managed to prove the result, but my working had been rather sketchy and difficult to put back together later.

In order to put the work on a more secure footing, I felt I should start with a more accurate relationship between this new variable and the caustic curves and managed, on paper, to obtain the next term in a series expansion. However, whilst preparing to discuss the work with Professor Jeffery and during the process of writing this lemma up on my computer, I realised two things:

1. If the solution is triple-valued within the region delimited by the caustic curves then there are exactly three unique characteristics which go through any given point. This implied an alternative to selecting two arbitrary characteristics and finding their intersection point: given an initial space time point it should be possible to prove that the characteristic equation has exactly three solutions for a point in a region bounded by the caustic curves. This could be shown if the characteristic equation was itself a cubic in terms of the space-time variables. Again, to derive this cubic equation was consistent with the catastrophe theory analysis I had done previously, so I felt I was on the right track. This would also overcome the problem of finding the third solution and proving that there were no others. (What I actaully had was *nearly* a cubic - a precise difinition of how near and how it affected using this method is beyond the scope of this article.)
2. What was required was an exact relationship between the caustic curves and the new space-time variable. I hoped to match this relationship exactly with the local boundary at which the cubic characteristic equation became multi-valued. This thought was only really half-formed in my mind.

I therefore switched direction and started looking at the characteristic equation for a fixed point in space. I managed to obtain this equation along with the exact solution for the caustic curves in terms of the new space-time variable. In the process of doing this, I reparametrized all the variables about the breaking point, putting a tilde on top of them.

(However, during the process of writing up this account I have realised that I have overly devalued the previous direction I had taken and now intend to combine both approaches.)

This is about where I had reached before the evening of 31 January 1995.

9.3 Account of the Evening's Work

(*Also written after 8 days.*)

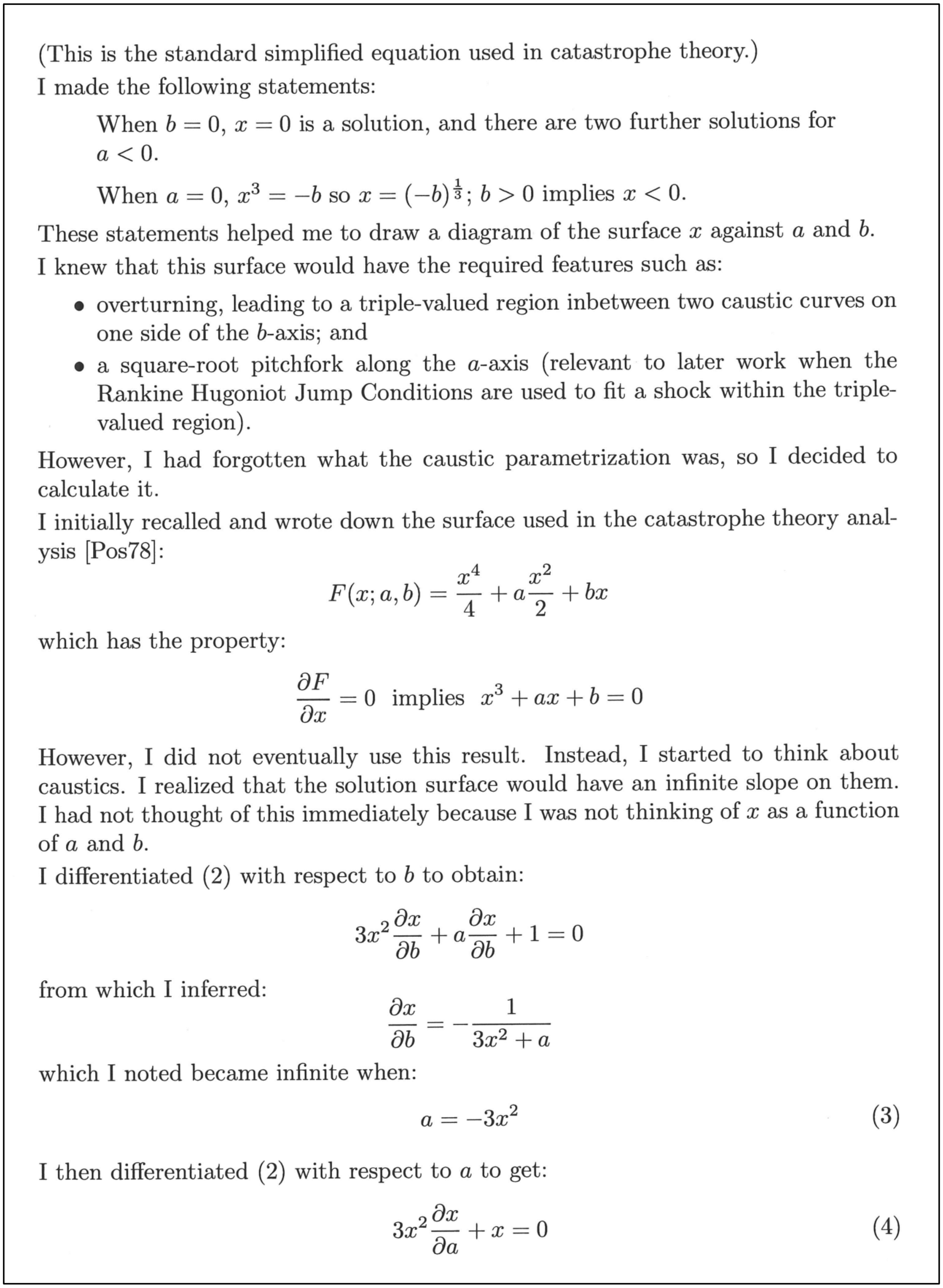
I started by writing down the reparametrized characteristic equation for a given space-time point:

$$\tilde{x} - c_0(\xi_B)\tilde{t} = c_0(\xi_B)\tilde{t}\tilde{\xi} - \frac{1 - c_0^{(1)}(\xi_B)\tilde{t}}{6c_0^{(1)}(\xi_B)} c_0^{(3)}(\xi_B + \hat{\xi})\tilde{\xi}^3 \qquad (1)$$

where $c_0(x)$ is the wave speed along the base line $t = 0$, $c_0^{(n)}$ is the nth derivative of c_0 and $\hat{\xi}$ is some value of ξ between 0 and $\tilde{\xi}$, given by a series expansion of one of the derivatives of c_0 (details to follow later).

I was thinking about this equation as a cubic in $\tilde{\xi}$. In order to have a clearer picture in my mind of what to do next, I decided to consider a model equation:

$$x^3 + ax + b = 0 \qquad (2)$$



(This is the standard simplified equation used in catastrophe theory.)

I made the following statements:

> When $b = 0$, $x = 0$ is a solution, and there are two further solutions for $a < 0$.
>
> When $a = 0$, $x^3 = -b$ so $x = (-b)^{\frac{1}{3}}$; $b > 0$ implies $x < 0$.

These statements helped me to draw a diagram of the surface x against a and b.

I knew that this surface would have the required features such as:

- overturning, leading to a triple-valued region inbetween two caustic curves on one side of the b-axis; and
- a square-root pitchfork along the a-axis (relevant to later work when the Rankine Hugoniot Jump Conditions are used to fit a shock within the triple-valued region).

However, I had forgotten what the caustic parametrization was, so I decided to calculate it.

I initially recalled and wrote down the surface used in the catastrophe theory analysis [Pos78]:

$$F(x; a, b) = \frac{x^4}{4} + a\frac{x^2}{2} + bx$$

which has the property:

$$\frac{\partial F}{\partial x} = 0 \text{ implies } x^3 + ax + b = 0$$

However, I did not eventually use this result. Instead, I started to think about caustics. I realized that the solution surface would have an infinite slope on them. I had not thought of this immediately because I was not thinking of x as a function of a and b.

I differentiated (2) with respect to b to obtain:

$$3x^2\frac{\partial x}{\partial b} + a\frac{\partial x}{\partial b} + 1 = 0$$

from which I inferred:

$$\frac{\partial x}{\partial b} = -\frac{1}{3x^2 + a}$$

which I noted became infinite when:

$$a = -3x^2 \tag{3}$$

I then differentiated (2) with respect to a to get:

$$3x^2\frac{\partial x}{\partial a} + x = 0 \tag{4}$$

implying:

$$\frac{\partial x}{\partial a} = -\frac{1}{3x}$$

which I noted was only infinite when $x = 0$. I could not understand why only the partial of x with respect to b gives an equation for the caustic (what I have later realised is that $x = 0$ does not necessarily imply this derivative is infinite because it is also a solution of (4); this means that the equation does not tell us anything about the infinite derivatives, something I still find puzzling, but I can now see how to investigate); taking a cross-section of the diagram parallel to the a-axis should work as well. I could only think that it was something to do with the fact that the caustic curves were initially parallel with the a-axis when they first formed and later only deviated asymptotically by a cubic, but I did not find this convincing - why should it affect cross-sections for finite values of b?

Leaving this problem unresolved, I carried on with my analysis, writing down a sequence of equations (I was trying to find the relationship between a and b on the caustics using (2) and (3)):

$$\begin{aligned} a &= 3x^2 \\ x^3 - 3x^3 + b &= 0 \\ b &= 2x^3 \end{aligned}$$

(having found a and b in terms of x, all I needed to do was eliminate x):

$$\begin{aligned} a^3 &= -27x^6 \\ b^2 &= 4x^6 \\ \text{so } \frac{b^2}{4} &= -\frac{a^3}{27} \\ \text{so } b &= \pm\frac{2}{3}\sqrt{-\frac{a^3}{3}} \end{aligned} \tag{5}$$

I then went back to the initial equation (1) with the objective of applying an anlogous process in order to obtain an equaion for the caustic curves.

I rewrote (1) using the new space-time variable $\tilde{z} = \tilde{x} - c_0(\xi_B)\tilde{t}$ and $f(\tilde{t})$ for the fractional part of the $\tilde{\xi}^3$ coefficient:

$$\tilde{z} = c_0(\xi_B)\tilde{t}\tilde{z} - f(\tilde{t})c_0^{(3)}(\xi_B + \hat{\xi})\tilde{\xi}^3 \tag{6}$$

I then proceeded with an analogous argument to that for the model equation:

$$\tilde{z} = 0 \quad \text{implies} \quad \tilde{\xi} = 0 \quad \text{or} \quad \tilde{\xi}^2 = \frac{c_0(\xi_B)\tilde{t}}{f(\tilde{t})c_0^{(3)}(\xi + \hat{\xi})}$$

I considered expanding the right hand side of this final equation as a power series in $\tilde{\xi}$ but instead chose to keep to the analogy and attmept to find $\tilde{\xi}$ as a function of $\tilde{z}$ and $\tilde{t}$. The analogous method would be to calculate $\dfrac{\partial\tilde{\xi}}{\partial\tilde{z}}$. However, in order to calculate this derivative, I realised that I needed an explicit term for $c_0^{(3)}(\xi_B + \hat{\xi})$ in terms of $\tilde{\xi}$.

I found the series expansion from which $\hat{\xi}$ was derived:

$$c_0(\xi_B + \tilde{\xi}) = c_0(\xi_B) + \tilde{\xi} c_0^{(1)}(\xi_B) + \int_0^{\tilde{\xi}} \frac{1}{2!}(\xi - \eta)^2 c_0^{(3)}(\xi_B + \eta) d\eta$$

(Note: there is no second derivative term in the expansion because $c_0(x)$ has an inflection point at $x = \xi_B$.)

The last term in the right-hand-side of this equation then matches with the term: $\dfrac{\tilde{\xi}^3}{3!} c_0^{(3)}(\xi_B + \hat{\xi})$

This meant that I was now able to rewrite (6) as:

$$\tilde{z} = c_0(\xi_B)\tilde{t}\tilde{\xi} - \frac{1 - c_0^{(1)}(\xi_B)\tilde{t}}{2c_0^{(1)}(\xi_B)} \int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 c_0^{(3)}(\tilde{\xi} + \eta) d\eta \tag{7}$$

I re-emphasised the fact that I was trying to find $\tilde{\xi}$ as a function of $\tilde{z}$ and $\tilde{t}$. I was now in a position to differentiate (7) with respect to $\tilde{z}$:

$$1 = c_0(\xi_B)\tilde{t}\frac{\partial\tilde{\xi}}{\partial\tilde{z}} - \left[\frac{1 - c_{)}^{(1)}(\xi_B)\tilde{t}}{2c_0^{(1)}(\xi_B)}\right] \frac{\partial}{\partial\tilde{z}} \int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 c_0^{(3)}(\xi_B + \eta) d\eta \tag{8}$$

Before proceeding, I was a bit concerned that the original equation I had used (1) was incorrect and sought supportive evidence from [Sam89] from which I obtained the equation:

$$\tilde{x} = \tilde{\xi} + c_0(\xi_B + \tilde{\xi}) \left\{ -\frac{1}{c_0^{(1)}(\xi_B)} + \tilde{t} \right\} + \frac{c_0(\xi_B)}{c_0^{(1)}(\xi_B)} \tag{9}$$

This seemed to be the right sort of thing so I didn't match the two equations exactly at this satge.

Instead, I switched back to calculating the partial derivative on the right-hand-side of (8):

$$\frac{\partial}{\partial\tilde{z}} \int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 c_0^{(3)}(\xi_B + \eta) d\eta = 2\frac{\partial\tilde{\xi}}{\partial\tilde{z}} \int_0^{\tilde{\xi}} (\tilde{\xi} - \eta) c_0^{(3)}(\xi_B + \eta) d\eta \tag{10}$$

I noted that the integral contribution vanishes as it is evaluated when $\eta = \tilde{\xi}$.

Combining (8) with (10) allowed me to state that the integral $\dfrac{\partial \tilde{\xi}}{\partial \tilde{z}}$ becomes infinite when:

$$c_0(\xi_B)\tilde{t} - \left[\frac{1 - c_0^{(1)}(\xi_B)\tilde{t}}{c_0^{(1)}(\xi_B)}\right] \int_0^{\tilde{\xi}} (\tilde{\xi} - \eta) c_0^{(3)}(\xi_B + \eta) d\eta = 0 \tag{11}$$

I wanted to rearrange this equation to make $\tilde{t}$ the subject. In order to simplify the working I decided to introduce an intermediate variable by defining:

$$\int_0^{\tilde{\xi}} (\tilde{\xi} - \eta) c_0^{(3)}(\xi_B + \eta) d\eta = \tilde{\xi}^2 I(\tilde{\xi}) \tag{12}$$

(The introduction of the $\tilde{\xi}^2$ term was to ensure that the integral $I(\tilde{\xi})$ was of the right order.)

Using this definition, I inferred that:

$$\tilde{t}\left[c_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right] = \frac{1}{c_0^{(1)}(\xi_B)} \tilde{\xi}^2 I(\tilde{\xi}) \tag{13}$$

which implied:

$$\tilde{t} = \frac{\tilde{\xi}^2 I(\tilde{\xi})}{c_0^{(1)}(\xi_B)\left[c_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right]} \tag{14}$$

I then considered whether I should attmept to eliminate $\tilde{\xi}$ from (7) by inverting (14). I envisaged doing this by expanding (14) as a power series in $\tilde{\xi}$ and back-substituting. I thought a suitable series would be of the form:

$$\tilde{\xi} = \alpha \tilde{t}^{\frac{1}{2}} + \beta \tilde{t} + \gamma \tilde{t}^{\frac{3}{2}} + \delta \tilde{t}^2 + \dots \tag{15}$$

In order to achieve this, I would have to find $I(\tilde{\xi})$ as a power series in $\tilde{t}$. I noted that this process should yield the same solution as the caustic parametrization equation (9) which I now rewrote as:

$$\tilde{z} = \tilde{\xi} - \frac{c_0(\xi_B + \tilde{\xi}) - c_0(\xi_B)}{c_0^{(1)}(\xi_B + \tilde{\xi})} \tag{16}$$

9.4 Review

(Notes made about three weeks later and written up after two and a half months.)

In trying to find a more explicit relationship between $\tilde{t}$ and $\tilde{\xi}$, I found I had made a mistake in equation (13): the term $c_0(\xi_B)$ in the first bracket should have been

$c_0^{(1)}(\xi_B)$. My mistake became evident when I tried to calculate the sign of $\tilde{t}$. Although I could have confirmed this with a dimensional analysis, I decided to make completely sure by going back to the parametric definition of the caustic curves. I applied the partial derivative method to the original characteristic equation as in the above analysis. This was an improvement over the previous method I had used which had involved calculating the equations of the caustics using neighbouring characteristics.

This gave me the symmetrical relationship:

$$\tilde{t} = \frac{c_0^{(1)}(\xi_B + \tilde{\xi}) - c_0^{(1)}(\xi_B)}{c_0^{(1)}(\xi_B) c_0^{(1)}(\xi_B + \tilde{\xi})} \tag{17}$$

This confimed my mistake (I later arrived at this equation via a different analysis) but also gave me some confidence about the internal consistency of the original analysis.

I was not completely happy about the reparametrized characteristic equation (1) and derived it again from first principles. As this is just a reparametrization of the original characteristic equation, I assumed that taking partial derivatives of the reparametrized chracteristic equation with respect to $\tilde{t}$ and $\tilde{x}$ would yield the same equation as (17). However, by analogy with the above working, I was not clear as to whether $\left.\frac{\partial \tilde{\xi}}{\partial \tilde{t}}\right|_{\tilde{z}}$ would become infinite on the edge of the solution surface. I was still not happy with the analysis of the model equation. I went back to it and found a mistake in (4) - I had left out one of the terms in differentiating the product ax with respect to a. My judgement was telling me something was wrong but I hadn't found the mistake at the time.

I am still not completely clear about the connection between infinite value of the derivatives on the edge of the characteristic surface (1) and a triple-valued region. I feel the answer lies in looking more closely at equation (17). I have had a couple of further attempts, but concede that I may need to resort (at least initially) to a catastrophe theory analysis, giving an existential result, rather than a constructive one.

Another thought I have had (off and on) is that the leading order power relationships between $\tilde{x}$, $\tilde{t}$, $\tilde{z}$ and $\tilde{\xi}$ are also suggested by the model equation analysis of the caustic curves. This is what I was doing in part in (15).

References

[Pos78] Poston, Tim and Stewart, Ian N. (1978). *Catastrophe Theory and its Applications.* London: Pitman.

[Sam89] Samuels, Peter (1989). *Shock Behaviour and Diffusion.* Technical Report 12/89, Mathematics Department, University of Reading.

Short term memory

Should be $C_0^{(1)}(\xi_B)$

31/1/95

A slightly deformed cubic equation.

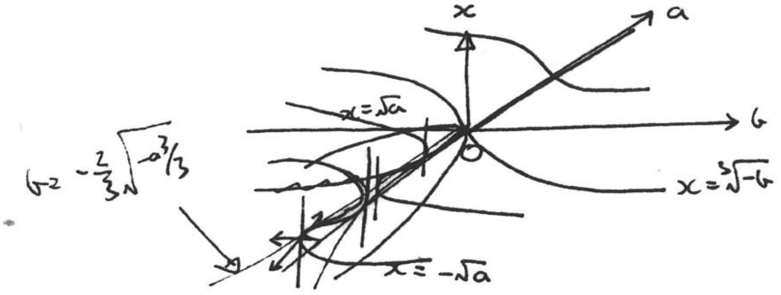
①

Objective/motivation:

$$\tilde{x} - C_0(\xi_B)\tilde{t} = \boxed{C_0(\xi_B)}\tilde{t}\,\tilde{\xi} - \frac{1 - C_0^{(1)}(\xi_B)\tilde{t}}{6C_0^{(1)}(\xi_B)}\, C_0^{(3)}(\xi_B + \hat{\xi})\,\tilde{\xi}^3$$

Consider the equation $x^3 + ax + b = 0$

When $b=0$, $x=0$ is a solution, 2 further solutions for $a<0$



$a=0$, $x^3 = -b$ $x = \sqrt[3]{-b}$ $b>0 \Rightarrow x<0$

$F(x;a,b) = \frac{x^4}{4} + a\frac{x^2}{2} + bx$ Not used.

$\frac{\partial F}{\partial x} = 0 \Rightarrow x^3 + ax + b = 0$

Caustics: Surface $x^3 + ax + b = 0$ has a repeated solution.

Consider x as a function of a & b

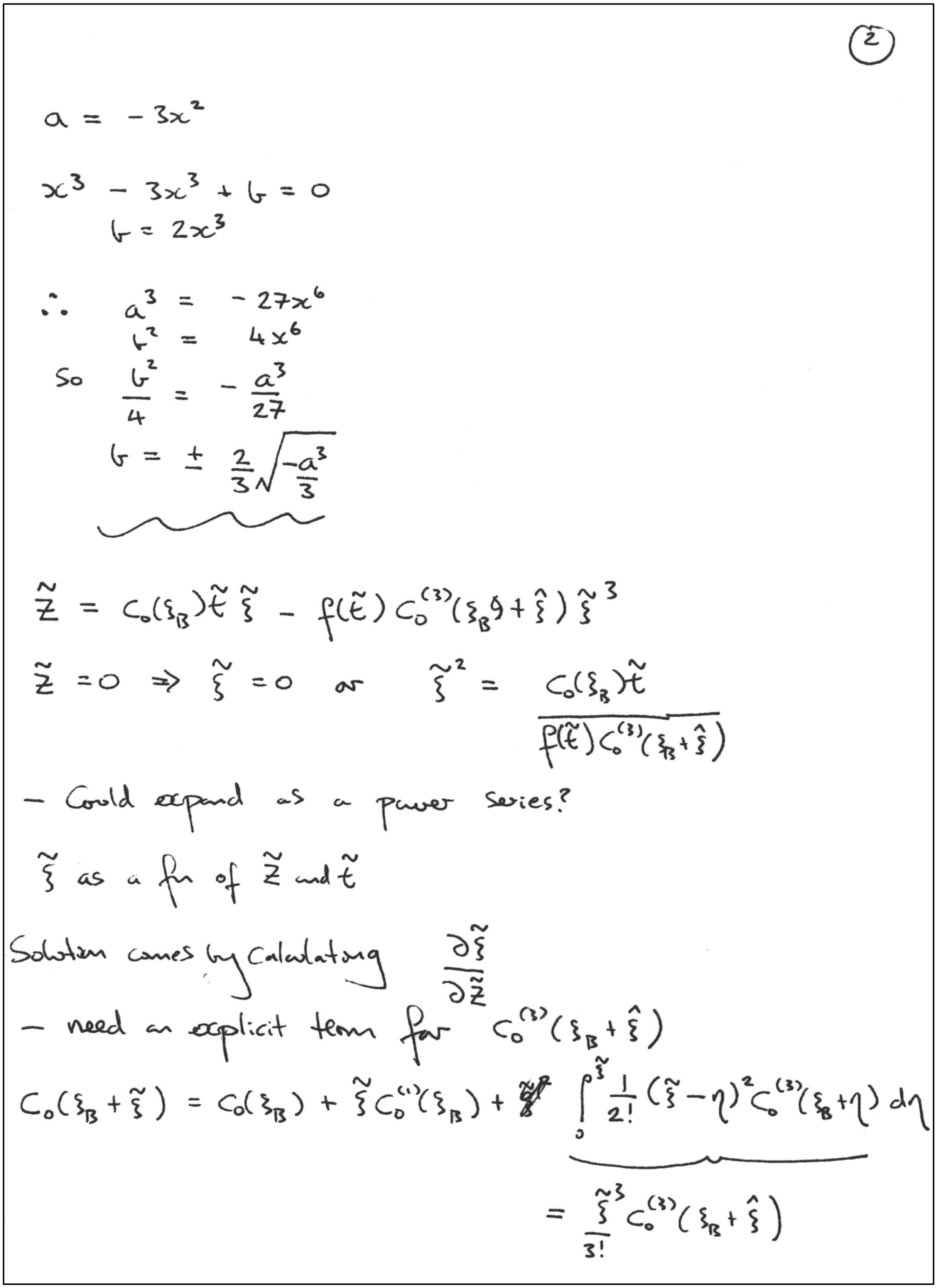
$$3x^2\frac{\partial x}{\partial b} + a\frac{\partial x}{\partial b} + 1 = 0$$

I dont understand why $\partial x/\partial b$ gives the right answer – a X-section || a-axise should work as well.

$$\frac{\partial x}{\partial b} = -\frac{1}{3x^2+a}$$ – infinite when $a = -3x^2$

& $3x^2\frac{\partial x}{\partial a} + x = 0 \Rightarrow \frac{\partial x}{\partial a} = -\frac{1}{3x}$ OR $x=0$

– infinite when $x=0$ only. – NO INFO.



②

$$a = -3x^2$$

$$x^3 - 3x^3 + b = 0$$
$$b = 2x^3$$

$$\therefore \quad a^3 = -27x^6$$
$$b^2 = 4x^6$$

So $\dfrac{b^2}{4} = -\dfrac{a^3}{27}$

$$b = \pm \frac{2}{3}\sqrt{-\frac{a^3}{3}}$$

* * *

$$\tilde{Z} = C_0(\xi_B)\tilde{t}\,\tilde{\xi} - f(\tilde{t})\,C_0^{(3)}(\xi_B + \hat{\xi})\,\tilde{\xi}^3$$

$$\tilde{Z} = 0 \Rightarrow \tilde{\xi} = 0 \quad \text{or} \quad \tilde{\xi}^2 = \frac{C_0(\xi_B)\tilde{t}}{f(\tilde{t})\,C_0^{(3)}(\xi_B + \hat{\xi})}$$

– Could expand as a power series?

$\tilde{\xi}$ as a fn of $\tilde{Z}$ and $\tilde{t}$

Solution comes by Calculating $\dfrac{\partial \tilde{\xi}}{\partial \tilde{Z}}$

– need an explicit term for $C_0^{(3)}(\xi_B + \hat{\xi})$

$$C_0(\xi_B + \tilde{\xi}) = C_0(\xi_B) + \tilde{\xi}\,C_0^{(1)}(\xi_B) + \cancel{\tilde{\xi}^2}\ \underbrace{\int_0^{\tilde{\xi}} \frac{1}{2!}(\tilde{\xi} - \eta)^2 C_0^{(3)}(\xi_B + \eta)\,d\eta}$$

$$= \frac{\tilde{\xi}^3}{3!}\,C_0^{(3)}(\xi_B + \hat{\xi})$$

③

$$\tilde{z} = C_0(\xi_B)\tilde{t}\,\tilde{\xi} - \frac{1 - C_0^{(1)}(\xi_B)\tilde{t}}{2C_0^{(1)}(\xi_B)} \int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)^2 C_0^{(3)}(\xi_B+\eta)\,d\eta \qquad Ⓐ$$

$$\tilde{\xi} = \tilde{\xi}(\tilde{z}, \tilde{t})$$

$$1 = C_0(\xi_B)\tilde{t}\,\frac{\partial\tilde{\xi}}{\partial\tilde{z}} - \left[\frac{1 - C_0^{(1)}(\xi_B)\tilde{t}}{2C_0^{(1)}(\xi_B)}\right] \frac{\partial}{\partial\tilde{z}} \int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)^2 C_0^{(3)}(\xi_B+\eta)\,d\eta$$

$$\tilde{x} = \tilde{\xi} + C_0(\xi_B+\tilde{\xi})\left\{-\frac{1}{C_0^{(1)}(\xi_B)} + \tilde{t}\right\} + \frac{C_0(\xi_B)}{C_0^{(1)}(\xi_B)} \qquad \text{(from report 4)}$$

$$\frac{\partial}{\partial\tilde{z}} \int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)^2 C_0^{(3)}(\xi_B+\eta)\,d\eta = 2\,\frac{\partial\tilde{\xi}}{\partial\tilde{z}} \int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)\, C_0^{(3)}(\xi_B+\eta)\,d\eta$$

as the integral term vanishes when $\eta = \tilde{\xi}$.

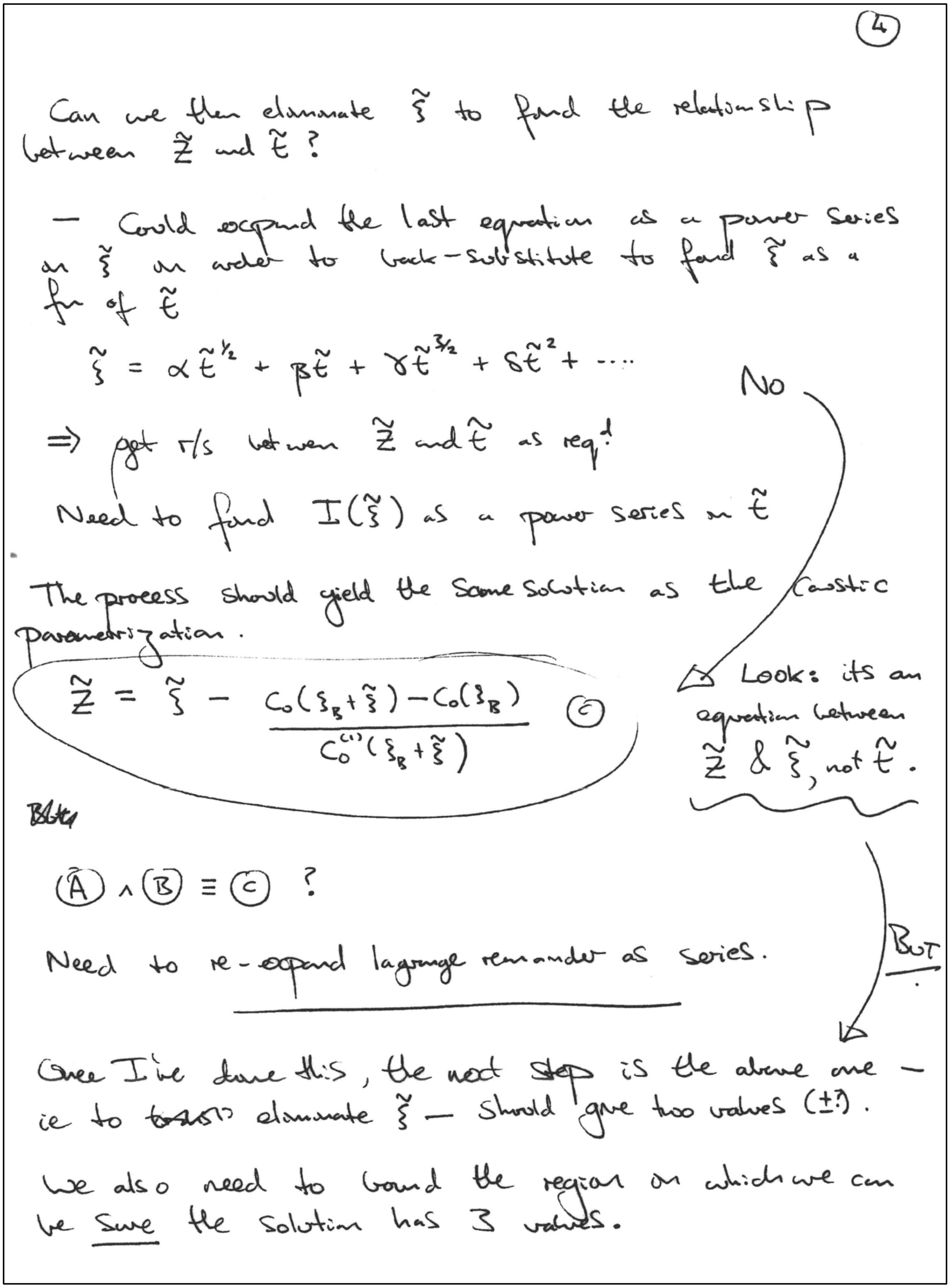
∴ vertical becomes infinite when:

$$C_0(\xi_B)\tilde{t} - \left[\frac{1 - C_0^{(1)}(\xi_B)\tilde{t}}{C_0^{(1)}(\xi_B)}\right] \int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)\, C_0^{(3)}(\xi_B+\eta)\,d\eta = 0 \qquad Ⓑ$$

Let $\displaystyle\int_0^{\tilde{\xi}} (\tilde{\xi}-\eta)\, C_0^{(3)}(\xi_B+\eta)\,d\eta = \tilde{\xi}^2 I(\tilde{\xi})$

So $\displaystyle\tilde{t}\left[C_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right] = \frac{1}{C_0^{(1)}(\xi_B)}\,\tilde{\xi}^2 I(\tilde{\xi})$

$$\tilde{t} = \frac{\tilde{\xi}^2 I(\tilde{\xi})}{C_0^{(1)}(\xi_B)\left[C_0(\xi_B) + \tilde{\xi}^2 I(\tilde{\xi})\right]}$$



④

Can we then elaminate $\tilde{\xi}$ to find the relationship between $\tilde{Z}$ and $\tilde{t}$?

— Could expand the last equation as a power series in $\tilde{\xi}$ in order to back-substitute to find $\tilde{\xi}$ as a fn of $\tilde{t}$

$$\tilde{\xi} = \alpha\tilde{t}^{1/2} + \beta\tilde{t} + \gamma\tilde{t}^{3/2} + \delta\tilde{t}^{2} + \cdots$$

No

⇒ get r/s between $\tilde{Z}$ and $\tilde{t}$ as req^{d}

Need to find $I(\tilde{\xi})$ as a power series in $\tilde{t}$

The process should yield the same solution as the caustic parametrization.

$$\tilde{Z} = \tilde{\xi} - \frac{C_0(\xi_B+\tilde{\xi}) - C_0(\xi_B)}{C_0^{(1)}(\xi_B+\tilde{\xi})} \quad ⓒ$$

Look: its an equation between $\tilde{Z}$ & $\tilde{\xi}$, not $\tilde{t}$.

~~[illegible]~~

Ⓐ ∧ Ⓑ ≡ ⓒ ?

Need to re-expand lagrange remainder as series.

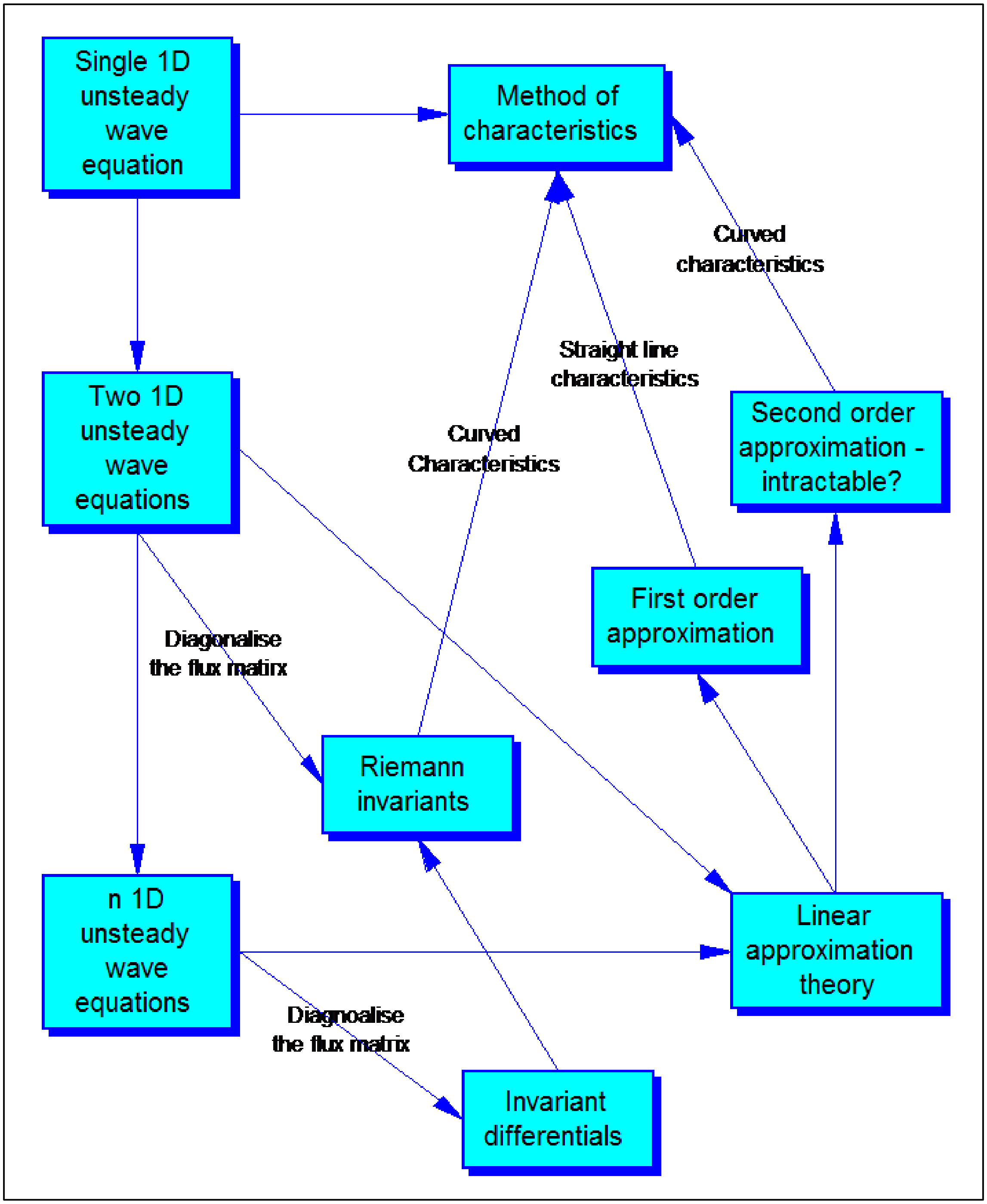
But.

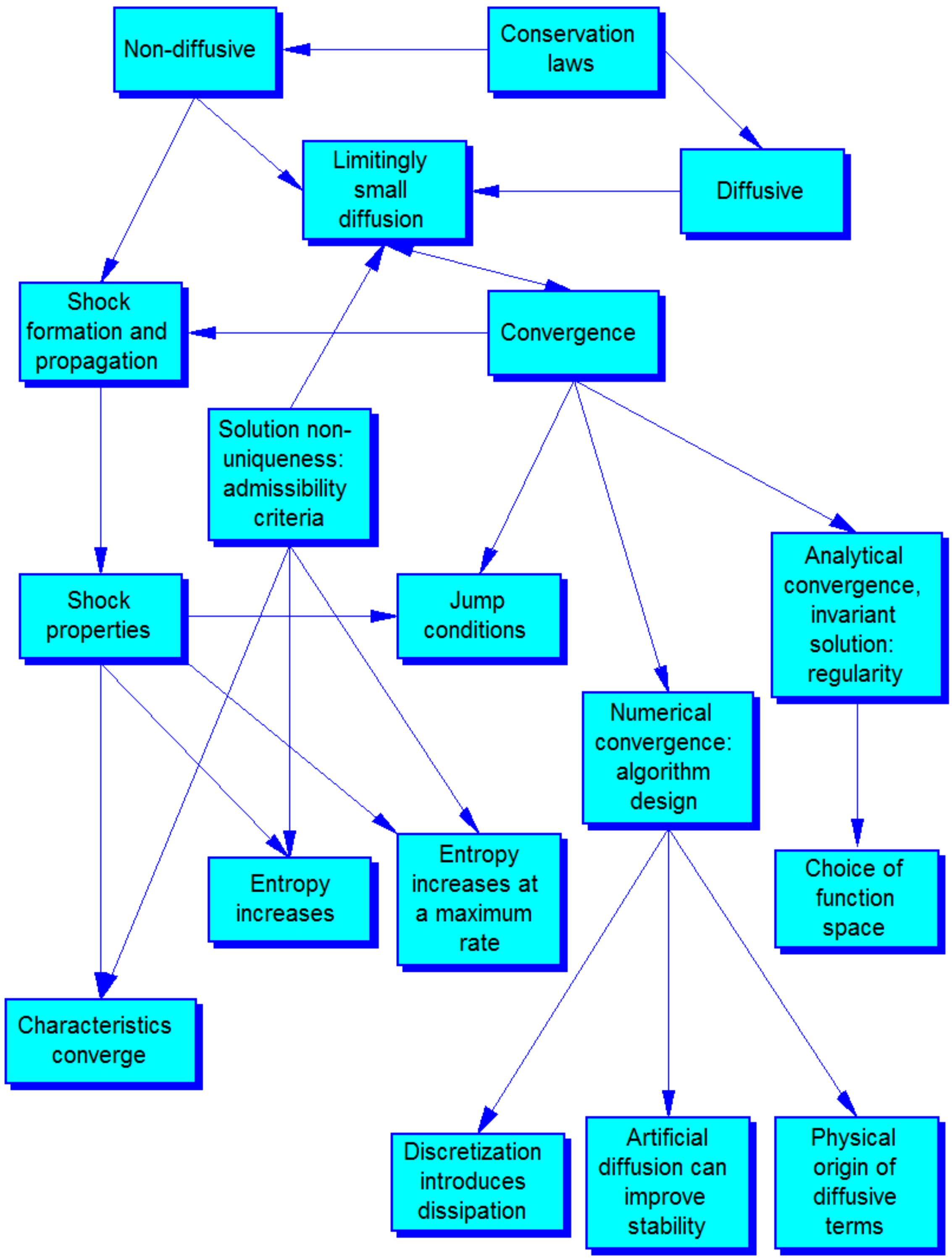
Once I've done this, the next step is the above one — ie to ~~[illegible]~~ elaminate $\tilde{\xi}$ — should give two values (±?).

We also need to bound the region in which we can be sure the solution has 3 values.

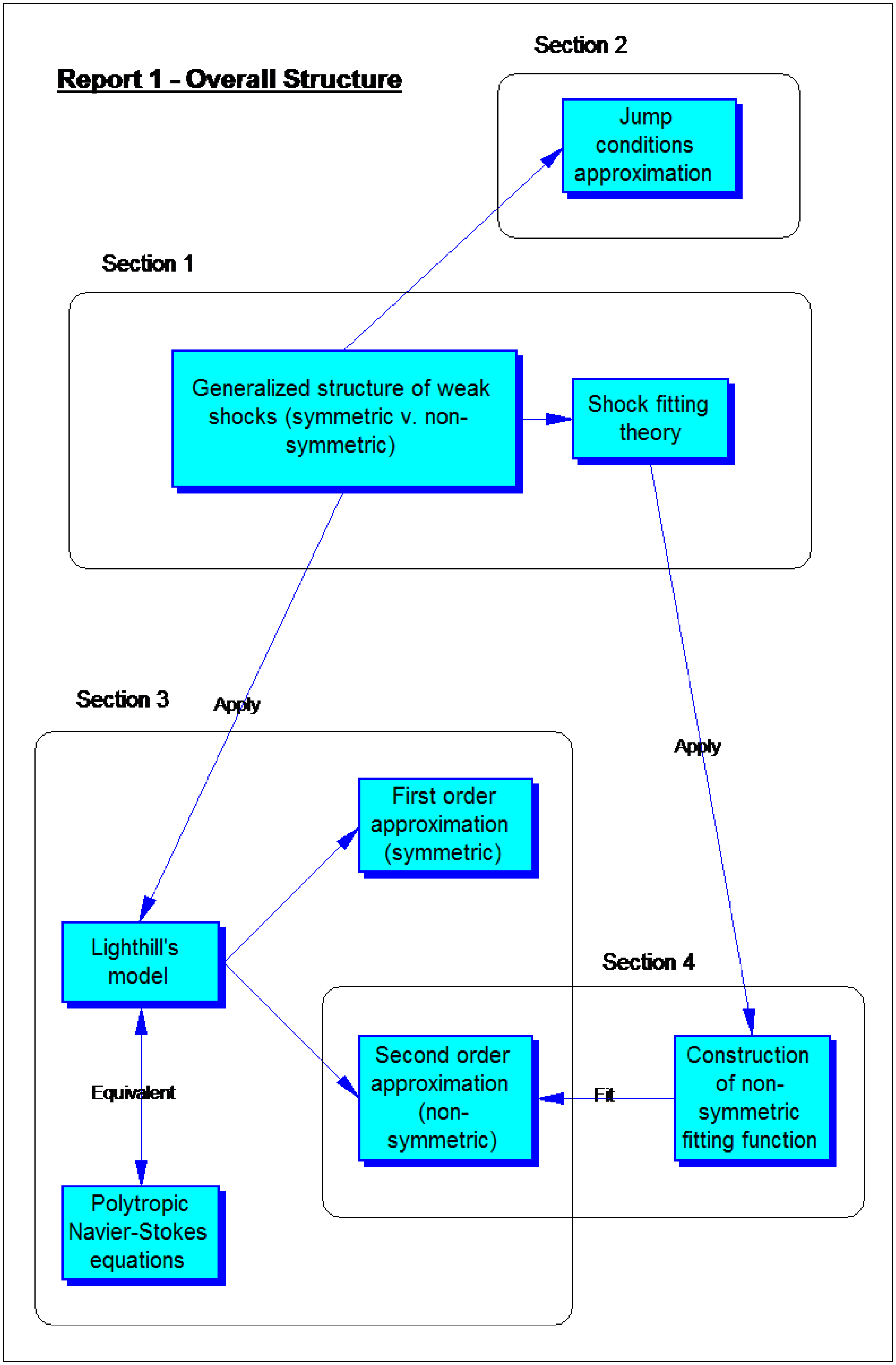
Appendix C

Examples of Mathematical Concept Maps

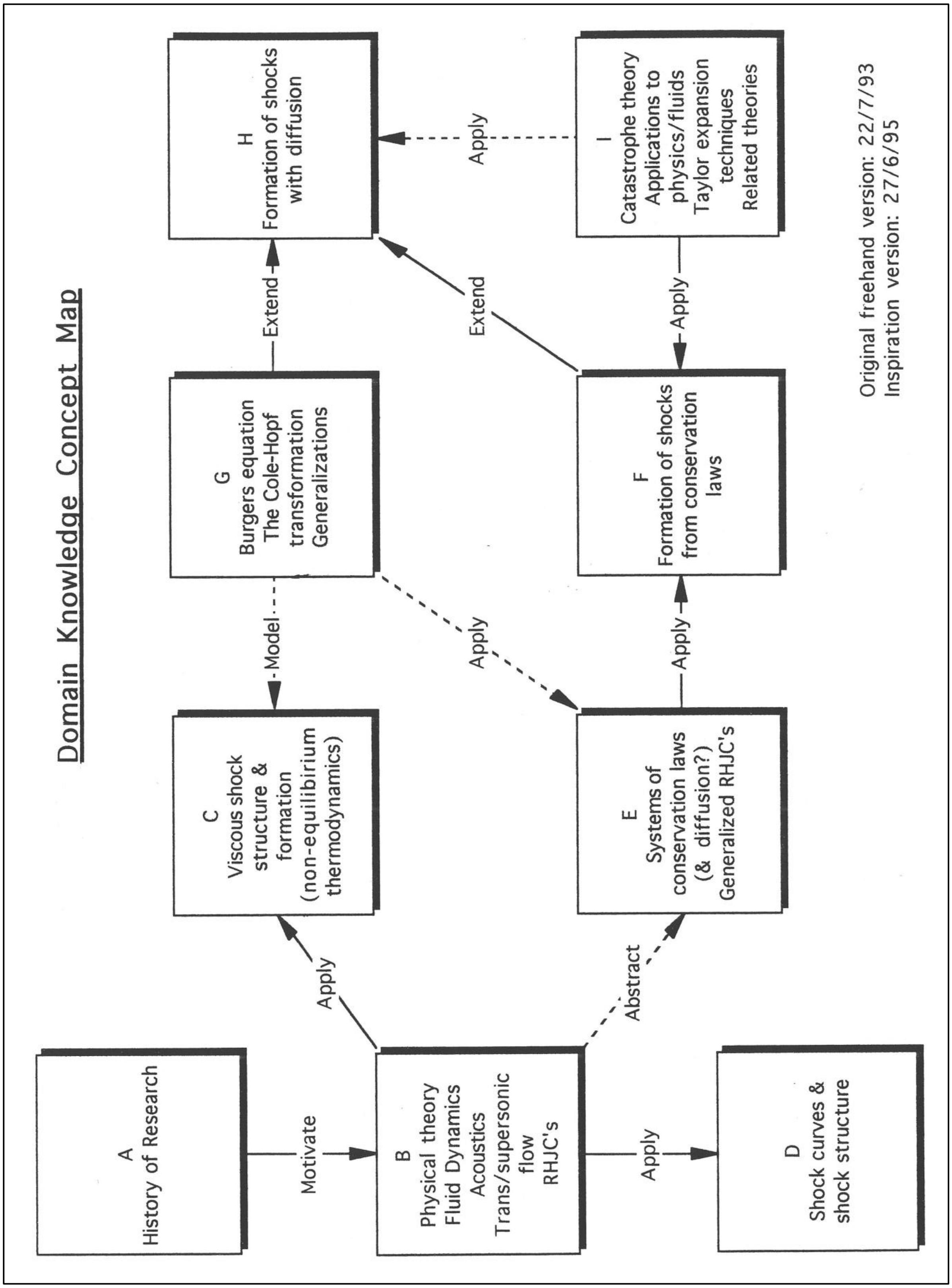




Mathematical Treatment of Conservation Laws with Limiting Diffusion

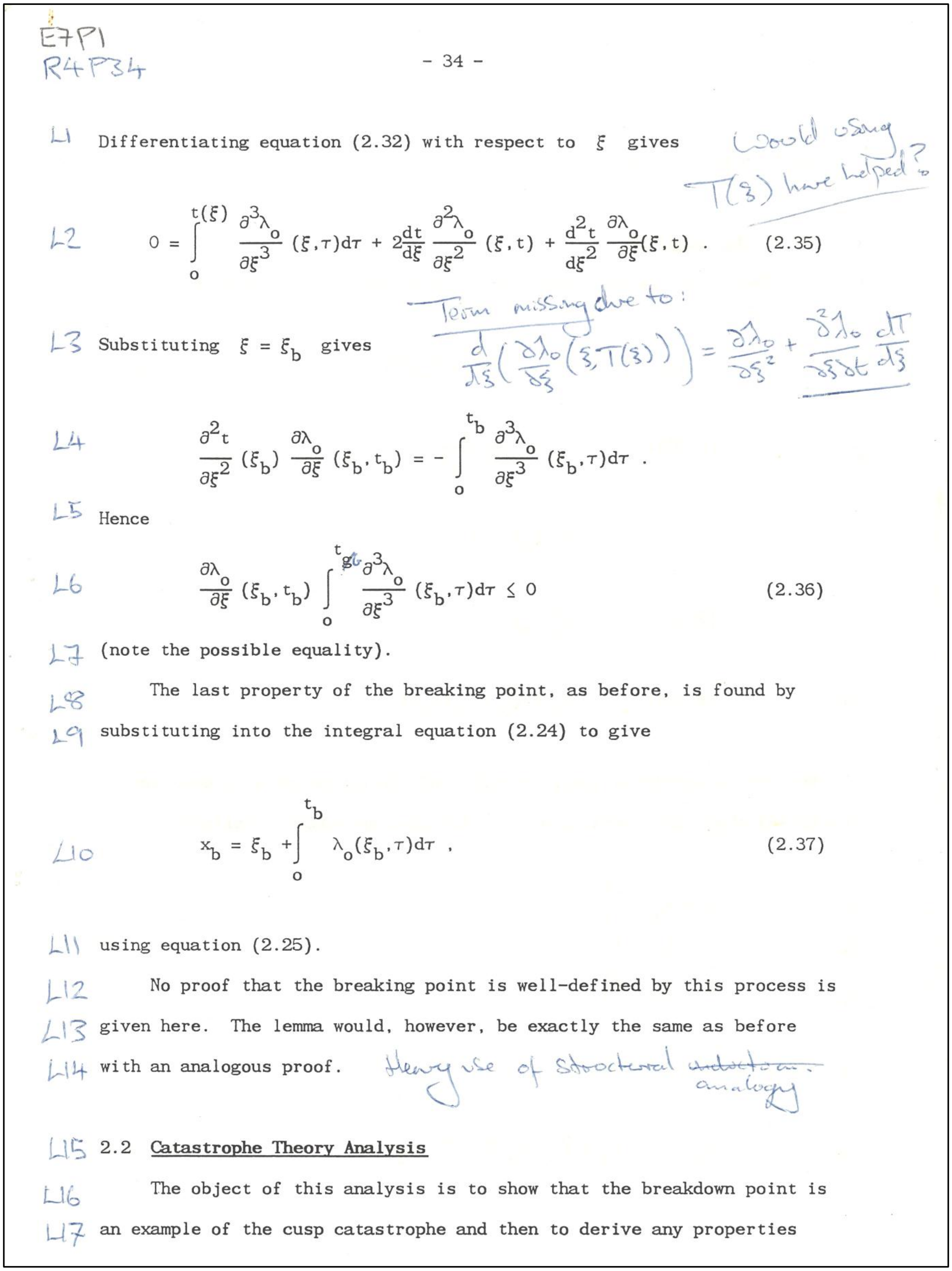


Report 1 - Overall Structure


Domain Knowledge Concept Map
A
History of Research
Motivate
B
Physical theory
Fluid Dynamics
Acoustics
Trans/supersonic
flow
RHJC's
Apply
D
Shock curves &
shock structure
Apply
C
Viscous shock
structure &
formation
(non-equilibirium
thermodynamics)
Abstract
E
Systems of
conservation laws
(& diffusion?)
Generalized RHJC's
Model
Apply
Apply
G
Burgers equation
The Cole-Hopf
transformation
Generalizations
F
Formation of shocks
from conservation
laws
Extend
Extend
Apply
H
Formation of shocks
with diffusion
Apply
I
Catastrophe theory
Applications to
physics/fluids
Taylor expansion
techniques
Related theories
Original freehand version: 22/7/93
Inspiration version: 27/6/95

Appendix D

Annotated Draft and Transcript of Entire Extract



E7P1
R4P34

- 34 -

L1 Differentiating equation (2.32) with respect to ξ gives

Would using $T(\xi)$ have helped?

L2
$$0 = \int_0^{t(\xi)} \frac{\partial^3\lambda_o}{\partial\xi^3}(\xi,\tau)d\tau + 2\frac{dt}{d\xi}\frac{\partial^2\lambda_o}{\partial\xi^2}(\xi,t) + \frac{d^2t}{d\xi^2}\frac{\partial\lambda_o}{\partial\xi}(\xi,t) \ . \qquad (2.35)$$

Term missing due to:

$$\frac{d}{d\xi}\left(\frac{\partial\lambda_o}{\partial\xi}(\xi,T(\xi))\right) = \frac{\partial\lambda_o}{\partial\xi^2} + \underline{\frac{\partial^2\lambda_o}{\partial\xi\partial t}\frac{dT}{d\xi}}$$

L3 Substituting $\xi = \xi_b$ gives

L4
$$\frac{\partial^2 t}{\partial\xi^2}(\xi_b)\ \frac{\partial\lambda_o}{\partial\xi}(\xi_b,t_b) = -\int_0^{t_b}\frac{\partial^3\lambda_o}{\partial\xi^3}(\xi_b,\tau)d\tau \ .$$

L5 Hence

L6
$$\frac{\partial\lambda_o}{\partial\xi}(\xi_b,t_b)\int_0^{t_{gb}}\frac{\partial^3\lambda_o}{\partial\xi^3}(\xi_b,\tau)d\tau \le 0 \qquad (2.36)$$

L7 (note the possible equality).

L8 The last property of the breaking point, as before, is found by
L9 substituting into the integral equation (2.24) to give

L10
$$x_b = \xi_b + \int_0^{t_b} \lambda_o(\xi_b,\tau)d\tau \ , \qquad (2.37)$$

L11 using equation (2.25).

L12 No proof that the breaking point is well-defined by this process is
L13 given here. The lemma would, however, be exactly the same as before
L14 with an analogous proof.

Heavy use of structural ~~induction~~ analogy

L15 2.2 <u>Catastrophe Theory Analysis</u>

L16 The object of this analysis is to show that the breakdown point is
L17 an example of the cusp catastrophe and then to derive any properties

E7P2
R4P35

QU

$\tilde{x}, \tilde{t}, \tilde{\xi}$

which naturally follow.

2.2.1 <u>One Equation</u>

Recalling the equations for the breaking point from §2.1.1:

$$(2.1_{a}),\ (2.3) \Rightarrow \quad x = \xi + \lambda_o(\xi)t\ , \qquad (2.38)$$

$$(2.10): \qquad t_b = -\frac{1}{\lambda_o'(\xi_b)}\ ,$$

$$(2.9): \qquad \lambda_o''(\xi_b) = 0\ ,$$

$$(2.12): \qquad \lambda_o'''(\xi_b) > 0\ ,$$

$$(2.13): \qquad x_b = \xi_b + \lambda_o(\xi_b)t_b\ .$$

We wish to perform a local rescaling of the characteristic equation (2.38) about the breaking point. Introducing rescaled variables

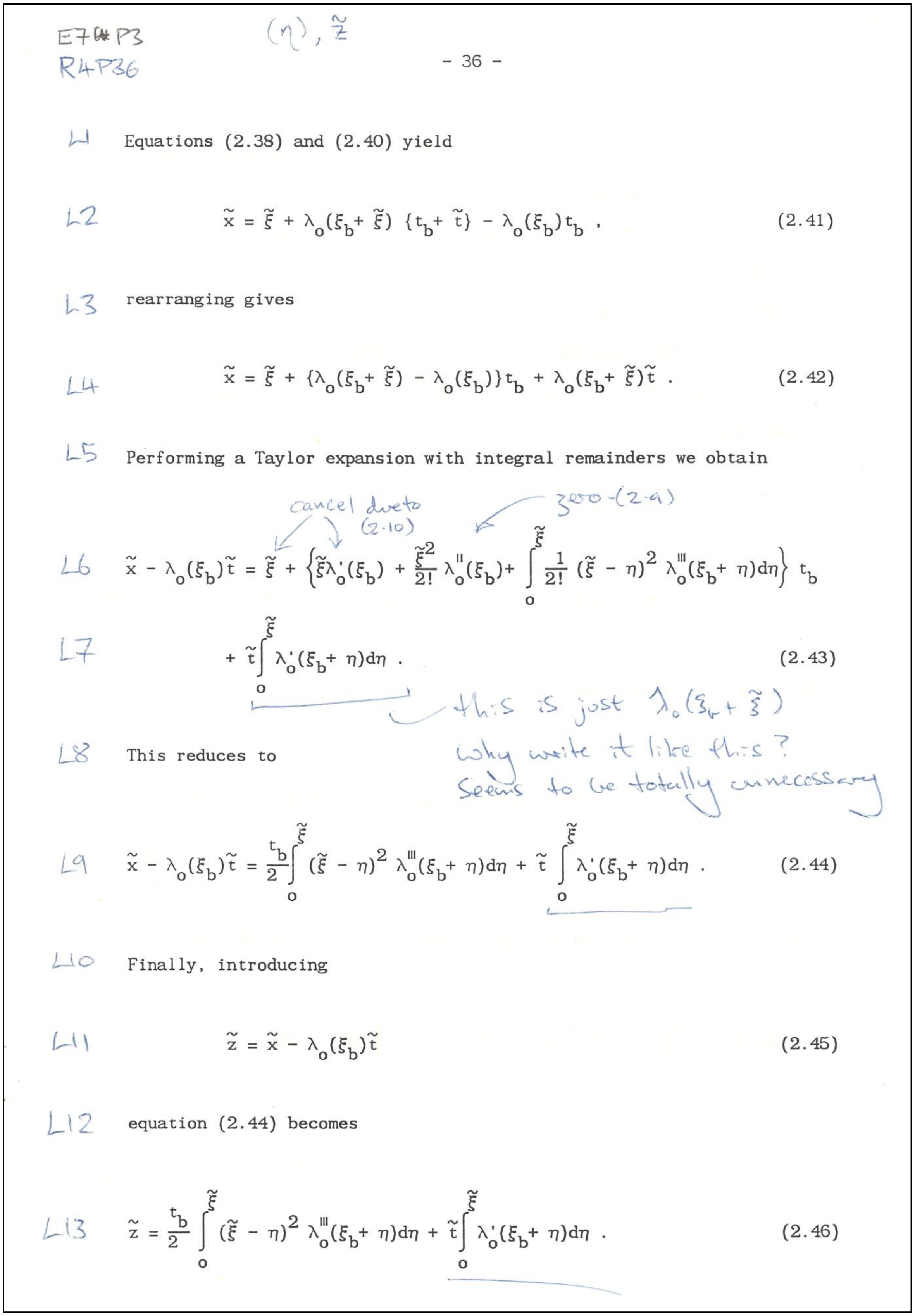
$$\left.\begin{aligned} x &= x_b + \tilde{x} \\ t &= t_b + \tilde{t} \\ \xi &= \xi_b + \tilde{\xi} \end{aligned}\right\}\ , \qquad (2.39)$$

Following Haberman

equation (2.38) becomes

$$x_b + \tilde{x} = \xi_b + \tilde{\xi} + \lambda_o(\xi_b + \tilde{\xi})\ \{t_b + \tilde{t}\}\ . \qquad (2.40)$$

Object – find simplified equation between $\tilde{x}$, $\tilde{t}$ & $\tilde{\xi}$



E7 ~~#~~ P3

R4P36

(η), z̃

- 36 -

L1 Equations (2.38) and (2.40) yield

L2 $$\tilde{x} = \tilde{\xi} + \lambda_o(\xi_b + \tilde{\xi})\ \{t_b + \tilde{t}\} - \lambda_o(\xi_b)t_b\ . \qquad (2.41)$$

L3 rearranging gives

L4 $$\tilde{x} = \tilde{\xi} + \{\lambda_o(\xi_b + \tilde{\xi}) - \lambda_o(\xi_b)\}t_b + \lambda_o(\xi_b + \tilde{\xi})\tilde{t}\ . \qquad (2.42)$$

L5 Performing a Taylor expansion with integral remainders we obtain

cancel due to (2.10)

zero (2.9)

L6 $$\tilde{x} - \lambda_o(\xi_b)\tilde{t} = \tilde{\xi} + \left\{\tilde{\xi}\lambda_o'(\xi_b) + \frac{\tilde{\xi}^2}{2!}\lambda_o''(\xi_b) + \int_0^{\tilde{\xi}} \frac{1}{2!}(\tilde{\xi} - \eta)^2 \lambda_o'''(\xi_b + \eta)d\eta\right\} t_b$$

L7 $$+ \underline{\tilde{t}\int_0^{\tilde{\xi}} \lambda_o'(\xi_b + \eta)d\eta}\ . \qquad (2.43)$$

this is just $\lambda_o(\xi_b + \tilde{\xi})$

why write it like this?

Seems to be totally unnecessary

L8 This reduces to

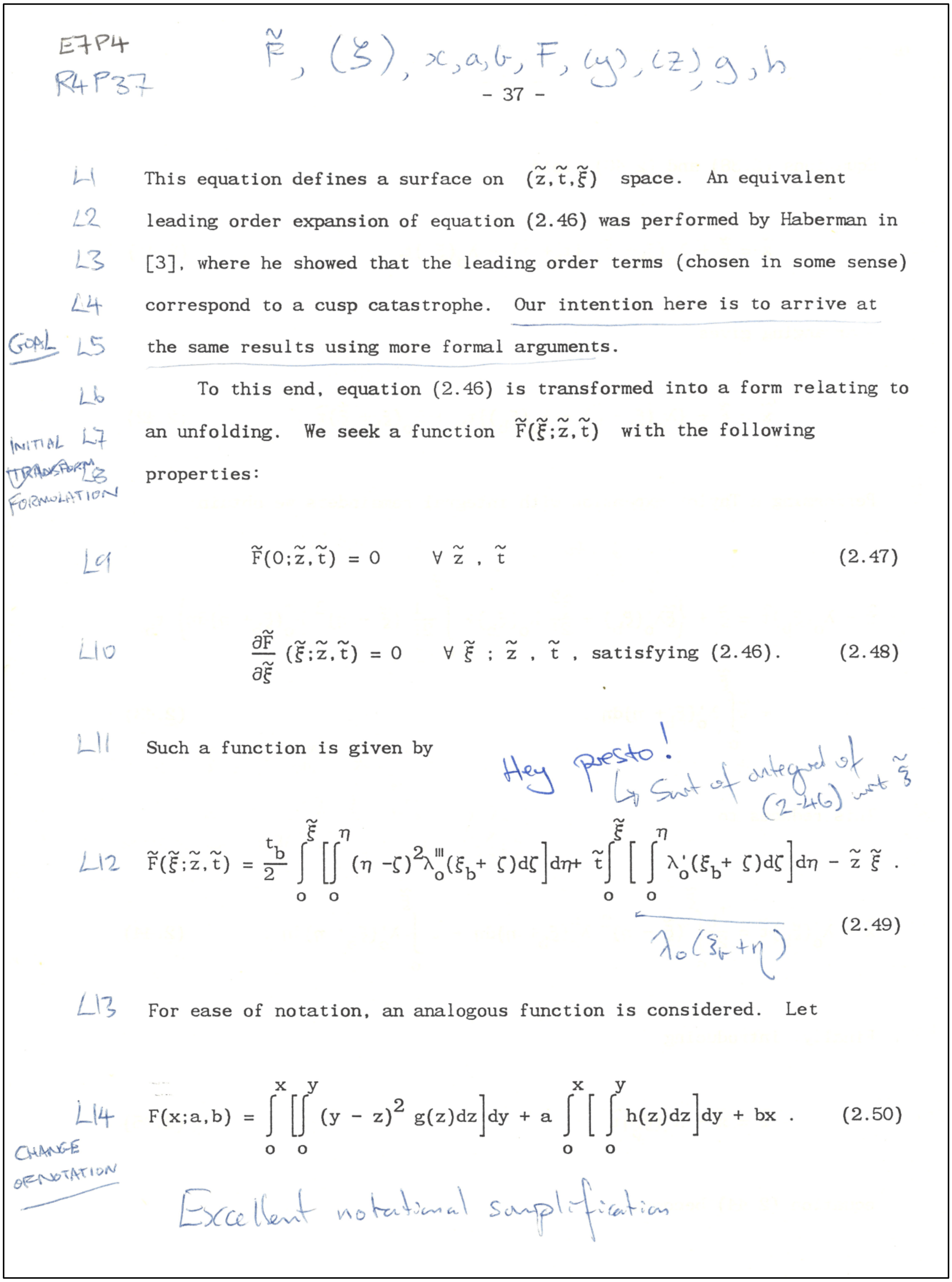
L9 $$\tilde{x} - \lambda_o(\xi_b)\tilde{t} = \frac{t_b}{2}\int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 \lambda_o'''(\xi_b + \eta)d\eta + \tilde{t}\ \underline{\int_0^{\tilde{\xi}} \lambda_o'(\xi_b + \eta)d\eta}\ . \qquad (2.44)$$

L10 Finally, introducing

L11 $$\tilde{z} = \tilde{x} - \lambda_o(\xi_b)\tilde{t} \qquad (2.45)$$

L12 equation (2.44) becomes

L13 $$\tilde{z} = \frac{t_b}{2}\int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 \lambda_o'''(\xi_b + \eta)d\eta + \underline{\tilde{t}\int_0^{\tilde{\xi}} \lambda_o'(\xi_b + \eta)d\eta}\ . \qquad (2.46)$$



E7P4
R4P37

$\tilde{F}$, (ξ), $x, a, b, F, (y), (z), g, h$

L1 This equation defines a surface on $(\tilde{z},\tilde{t},\tilde{\xi})$ space. An equivalent
L2 leading order expansion of equation (2.46) was performed by Haberman in
L3 [3], where he showed that the leading order terms (chosen in some sense)
L4 correspond to a cusp catastrophe. Our intention here is to arrive at
GOAL L5 the same results using more formal arguments.

L6 To this end, equation (2.46) is transformed into a form relating to
INITIAL TRANSFORM FORMULATION L7 an unfolding. We seek a function $\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t})$ with the following
L8 properties:

L9
$$\tilde{F}(0;\tilde{z},\tilde{t}) = 0 \qquad \forall\ \tilde{z}\ ,\ \tilde{t} \tag{2.47}$$

L10
$$\frac{\partial\tilde{F}}{\partial\tilde{\xi}}(\tilde{\xi};\tilde{z},\tilde{t}) = 0 \qquad \forall\ \tilde{\xi}\ ;\ \tilde{z}\ ,\ \tilde{t}\ ,\ \text{satisfying (2.46).} \tag{2.48}$$

L11 Such a function is given by

Hey presto!
↳ Sort of integral of (2-46) wrt $\tilde{\xi}$

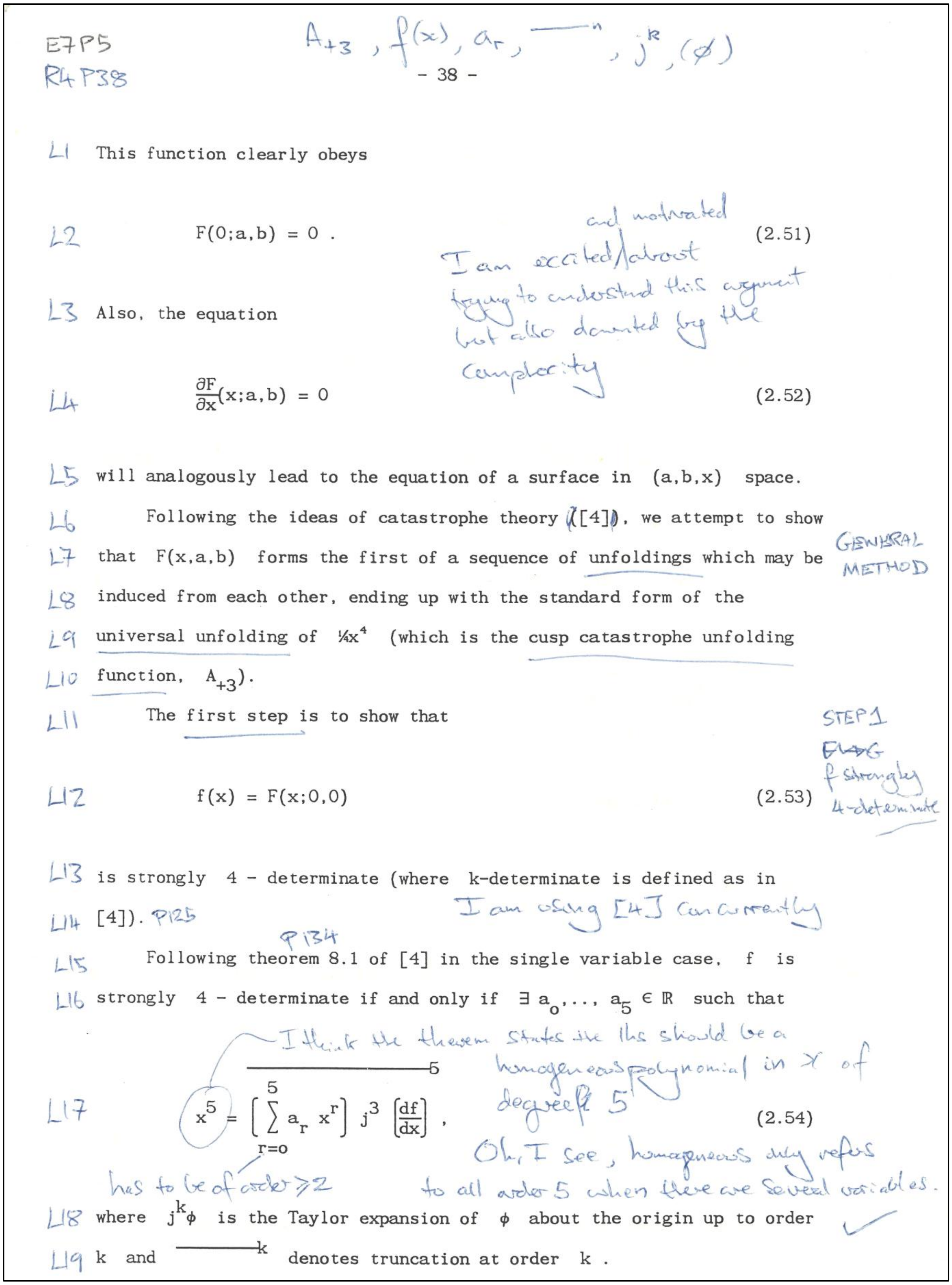
L12
$$\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t}) = \frac{t_b}{2}\int_0^{\tilde{\xi}}\left[\int_0^{\eta}(\eta-\zeta)^2\lambda_o'''(\xi_b+\zeta)d\zeta\right]d\eta + \tilde{t}\int_0^{\tilde{\xi}}\left[\underbrace{\int_0^{\eta}\lambda_o'(\xi_b+\zeta)d\zeta}_{\lambda_o(\xi_b+\eta)}\right]d\eta - \tilde{z}\,\tilde{\xi}\ . \tag{2.49}$$

L13 For ease of notation, an analogous function is considered. Let

L14
$$F(x;a,b) = \int_0^x\left[\int_0^y(y-z)^2\,g(z)dz\right]dy + a\int_0^x\left[\int_0^y h(z)dz\right]dy + bx\ . \tag{2.50}$$

CHANGE OF NOTATION

Excellent notational simplification



E7P5
R4P38

A_{+3} , $f(x)$, a_r, $\overline{}^{\,n}$, j^k, (ϕ)

L1 This function clearly obeys

L2 $$F(0;a,b) = 0 \;. \tag{2.51}$$

I am excited and motivated about trying to understand this argument but also daunted by the complexity

L3 Also, the equation

L4 $$\frac{\partial F}{\partial x}(x;a,b) = 0 \tag{2.52}$$

L5 will analogously lead to the equation of a surface in (a,b,x) space.

L6 Following the ideas of catastrophe theory ([4]), we attempt to show
L7 that F(x,a,b) forms the first of a sequence of unfoldings which may be
L8 induced from each other, ending up with the standard form of the
L9 universal unfolding of ¼x^4 (which is the cusp catastrophe unfolding
L10 function, A_{+3}).

GENERAL METHOD

L11 The first step is to show

STEP 1
FLAG
f strongly 4-determinate

L12 $$f(x) = F(x;0,0) \tag{2.53}$$

L13 is strongly 4 - determinate (where k-determinate is defined as in
L14 [4]). P125

I am using [4] concurrently

P134

L15 Following theorem 8.1 of [4] in the single variable case, f is
L16 strongly 4 - determinate if and only if $\exists\, a_o, \ldots, a_5 \in \mathbb{R}$ such that

I think the theorem states the lhs should be a homogeneous polynomial in x of degree 5

L17 $$x^5 = \overline{\left[\sum_{r=o}^{5} a_r\, x^r\right] j^3 \left[\frac{df}{dx}\right]}^{\,5} \;, \tag{2.54}$$

has to be of order ≥2

Oh, I see, homogeneous only refers to all order 5 when there are several variables. ✓

L18 where $j^k\phi$ is the Taylor expansion of ϕ about the origin up to order
L19 k and $\overline{}^{\,k}$ denotes truncation at order k .

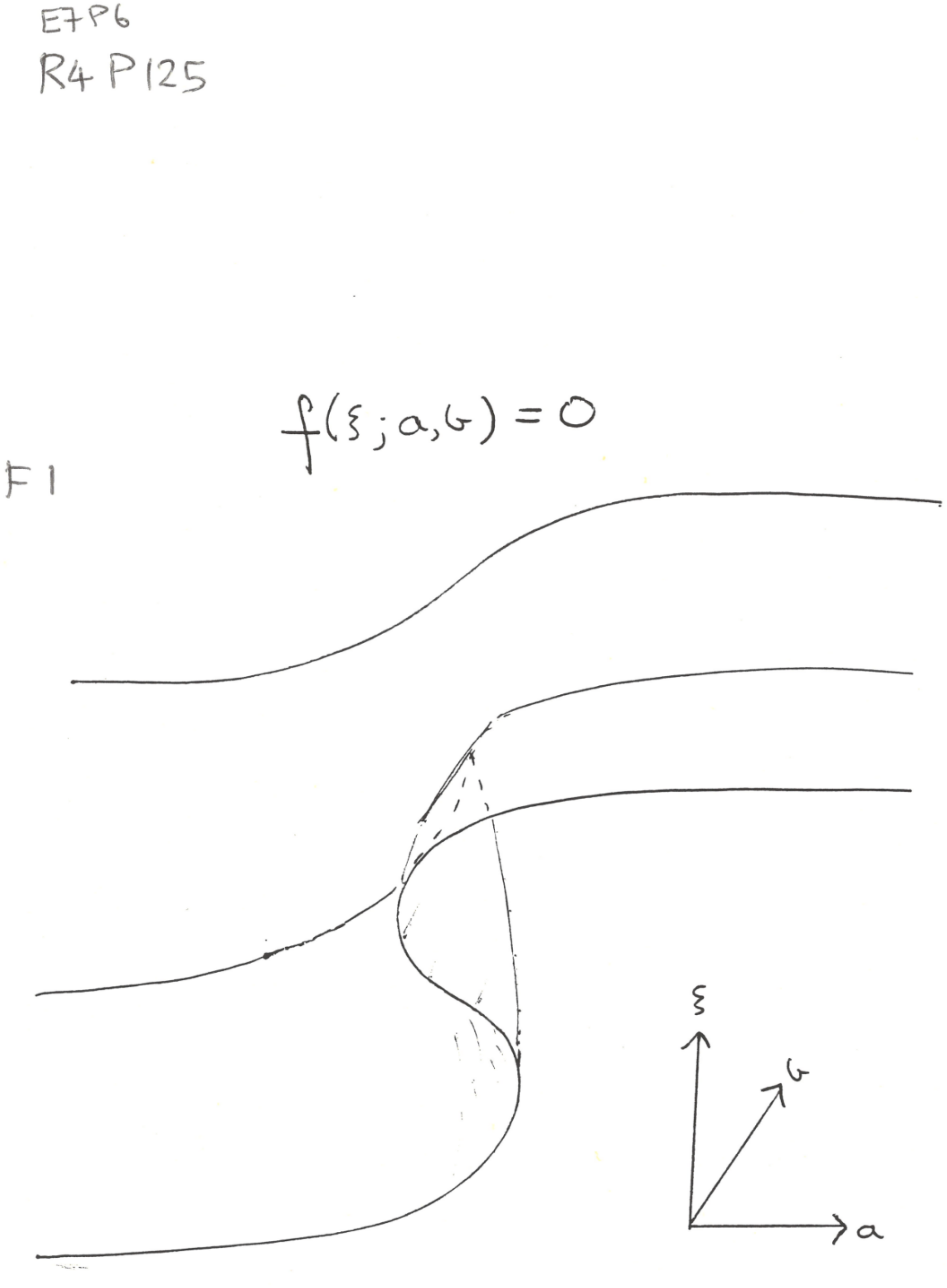


Figure 4.

E7P7
R4P39

g(o)

Equations (2.50) and (2.53) imply

$$f(x) = \int_o^x \left[\int_o^y (y-z)^2 \; g(z)dz \right] dy \; . \qquad (2.55)$$

Calculating successive derivatives of f gives

$$f'(x) = \int_o^x (x-y)^2 \; g(y) \; dy \; , \qquad (2.56)$$

$$f''(x) = \int_o^x 2(x-y) \; g(y) \; dy \; , \qquad (2.57)$$

$$f'''(x) = \int_o^x 2g(y) \; dy \; , \qquad (2.58)$$

$$f^{IV}(x) = 2g(x) \; . \qquad (2.59)$$

✓

Recalling the Taylor expansion for $f'(x)$,

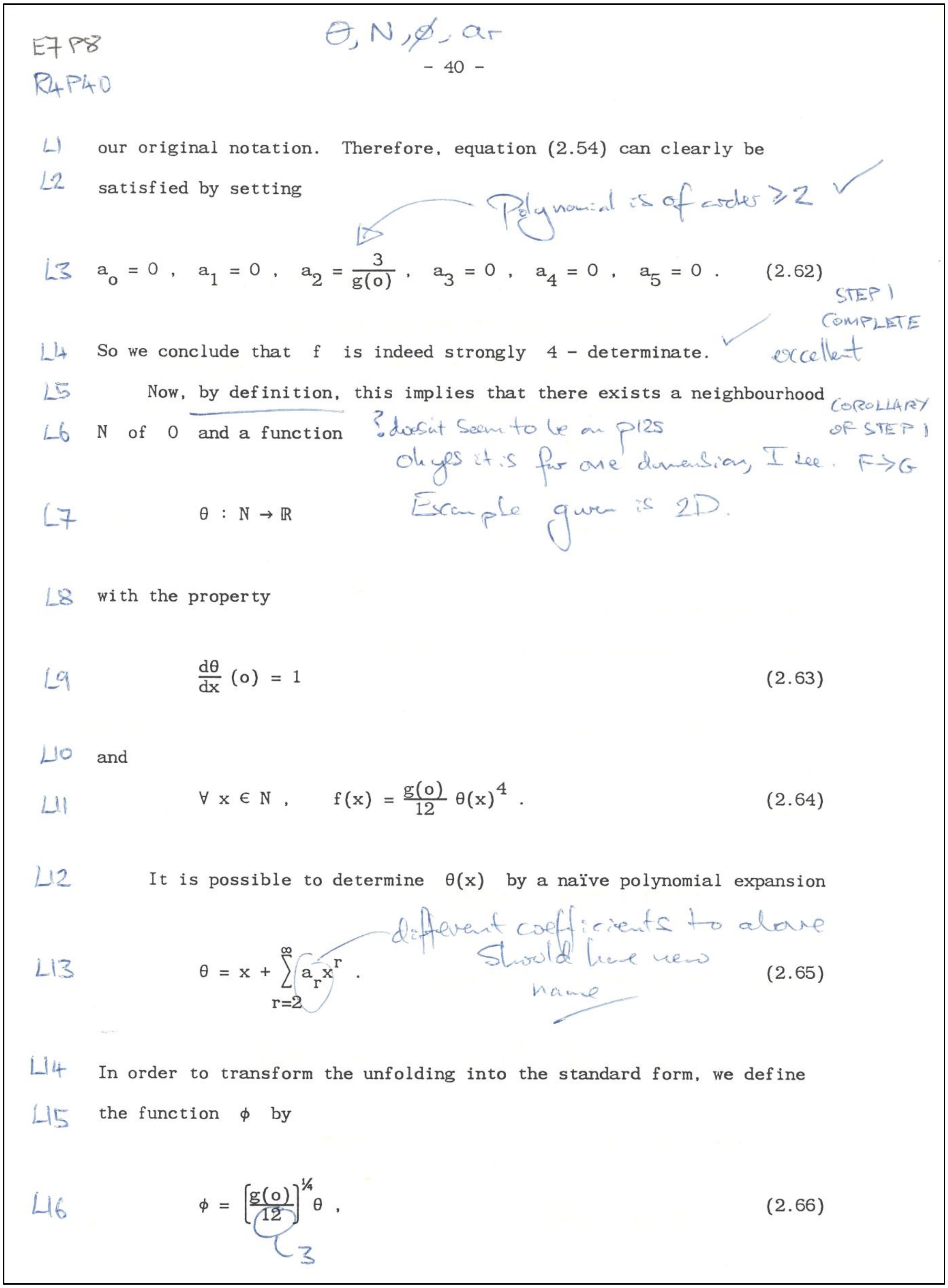
$$j^3 f'(x) = f'(o) + xf''(o) + \frac{x^2}{2!} f'''(o) + \frac{x^3}{3!} f^{IV}(o) \; . \qquad (2.60)$$

Therefore, we obtain

$$j^3 \; f'(x) = \frac{x^3}{3} g(o) \; . \qquad (2.61)$$

We may assume $g(o) > 0$ as this corresponds to equation (2.12) in

✓



E7P8

R4P40

θ, N, ϕ, a_r

- 40 -

L1 our original notation. Therefore, equation (2.54) can clearly be

L2 satisfied by setting

Polynomial is of order ≥ 2 ✓

L3 $$a_o = 0\ ,\quad a_1 = 0\ ,\quad a_2 = \frac{3}{g(o)}\ ,\quad a_3 = 0\ ,\quad a_4 = 0\ ,\quad a_5 = 0\ . \qquad (2.62)$$

STEP 1 COMPLETE

L4 So we conclude that f is indeed strongly 4 - determinate. ✓ excellent

L5 Now, by definition, this implies that there exists a neighbourhood

COROLLARY OF STEP 1

L6 N of 0 and a function

? doesn't seem to be on p125

Oh yes it is for one dimension, I see. F→G

Example given is 2D.

L7 $$\theta : N \to \mathbb{R}$$

L8 with the property

L9 $$\frac{d\theta}{dx}(o) = 1 \qquad (2.63)$$

L10 and

L11 $$\forall\ x \in N\ , \qquad f(x) = \frac{g(o)}{12}\,\theta(x)^4\ . \qquad (2.64)$$

L12 It is possible to determine $\theta(x)$ by a naïve polynomial expansion

different coefficients to above Should have new name

L13 $$\theta = x + \sum_{r=2}^{\infty} a_r x^r\ . \qquad (2.65)$$

L14 In order to transform the unfolding into the standard form, we define

L15 the function ϕ by

L16 $$\phi = \left[\frac{g(o)}{12}\right]^{1/4}\theta\ , \qquad (2.66)$$

3

E7 P9
R4 P41

$A, g'(0),$

so

$$f(x) = \frac{\phi(x)^4}{4} . \qquad (2.67)$$

might this cause a problem

It will be assumed that this function is invertible for small x and ϕ, and that the inverse function may be approximated by a finite Taylor series. It will turn out later that we require the quadratic term in this expansion. For simplicity, let us first consider the inversion of equation (2.65):

$$x(\theta) = \theta + A\theta^2 + 0(\theta)^3 \qquad (2.68)$$

$$x(\phi) = \left[\frac{3}{g(0)}\right]^{1/4} \phi + \dots$$

for some constant A.

But, differentiating equation (2.59) gives

$$f^V(x) = 2g'(x) . \qquad (2.69)$$

Therefore,

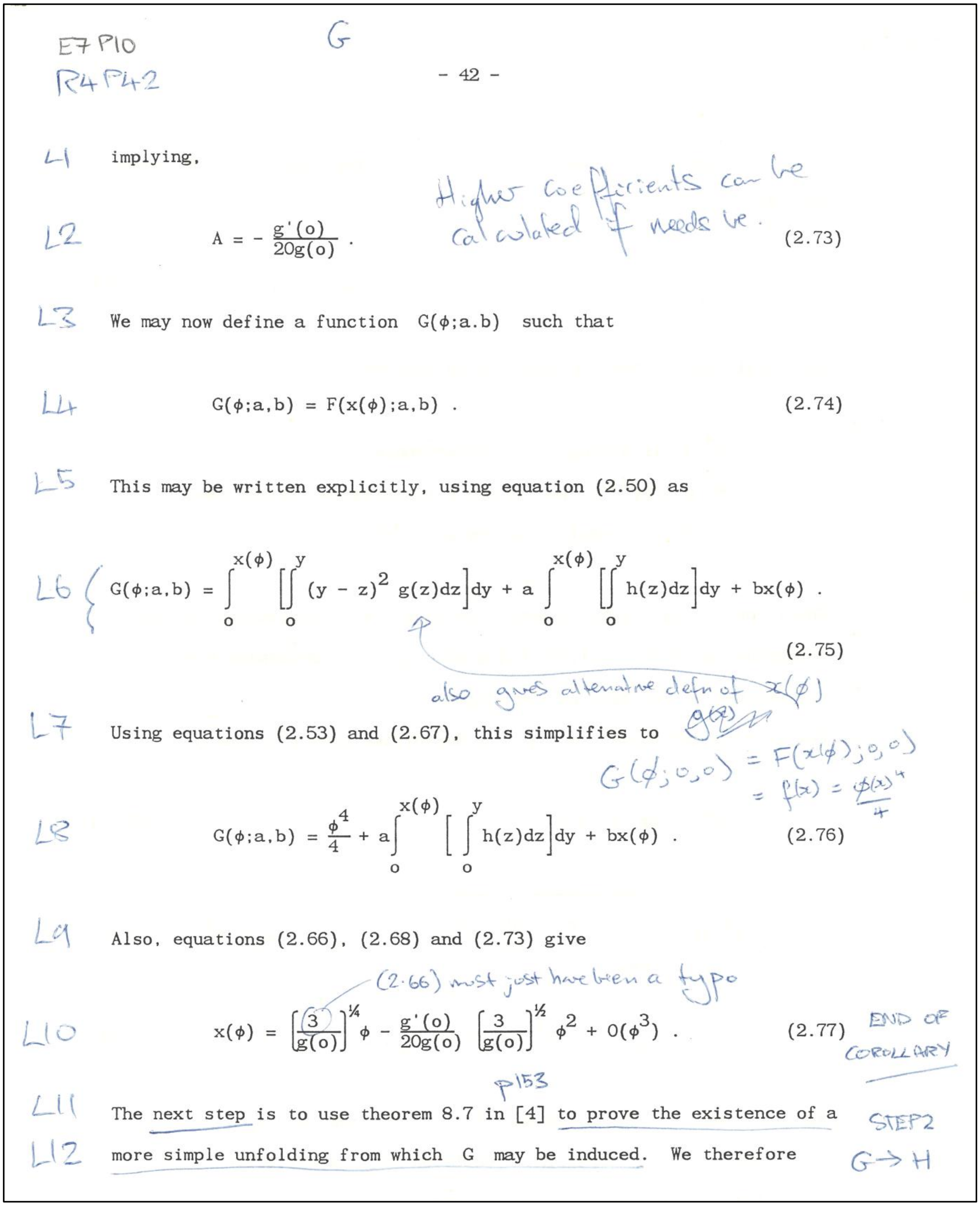
$$j^5 f(x) = \frac{g(o)}{12} x(\theta)^4 + \frac{g'(o)}{60} x(\theta)^5 . \qquad (2.70)$$

Substituting in equation (2.68) gives

$$f(x(\theta)) = \frac{g(o)}{12} \{\theta^4 + 4A\theta^5\} + \frac{g'(o)}{60} \theta^5 + 0(\theta^6) . \qquad (2.71)$$

Thus, equating terms of fifth order in θ gives

$$\frac{g(o)}{12} 4A + \frac{g'(o)}{60} = 0 , \qquad (2.72)$$



E7 P10
R4 P42

G

- 42 -

L1 implying,

L2
$$A = -\frac{g'(o)}{20g(o)} \ . \qquad (2.73)$$

Higher coefficients can be calculated if needs be.

L3 We may now define a function $G(\phi;a.b)$ such that

L4
$$G(\phi;a,b) = F(x(\phi);a,b) \ . \qquad (2.74)$$

L5 This may be written explicitly, using equation (2.50) as

L6
$$G(\phi;a,b) = \int_o^{x(\phi)} \left[\int_o^y (y-z)^2\, g(z)dz\right]dy + a\int_o^{x(\phi)} \left[\int_o^y h(z)dz\right]dy + bx(\phi) \ .$$

(2.75)

also gives alternative defn of $x(\phi)$

$g(x)$

L7 Using equations (2.53) and (2.67), this simplifies to

$G(\phi;0,0) = F(x(\phi);0,0)$
$= f(x) = \frac{\phi(x)^4}{4}$

L8
$$G(\phi;a,b) = \frac{\phi^4}{4} + a\int_o^{x(\phi)} \left[\int_o^y h(z)dz\right]dy + bx(\phi) \ . \qquad (2.76)$$

L9 Also, equations (2.66), (2.68) and (2.73) give

(2.66) must just have been a typo

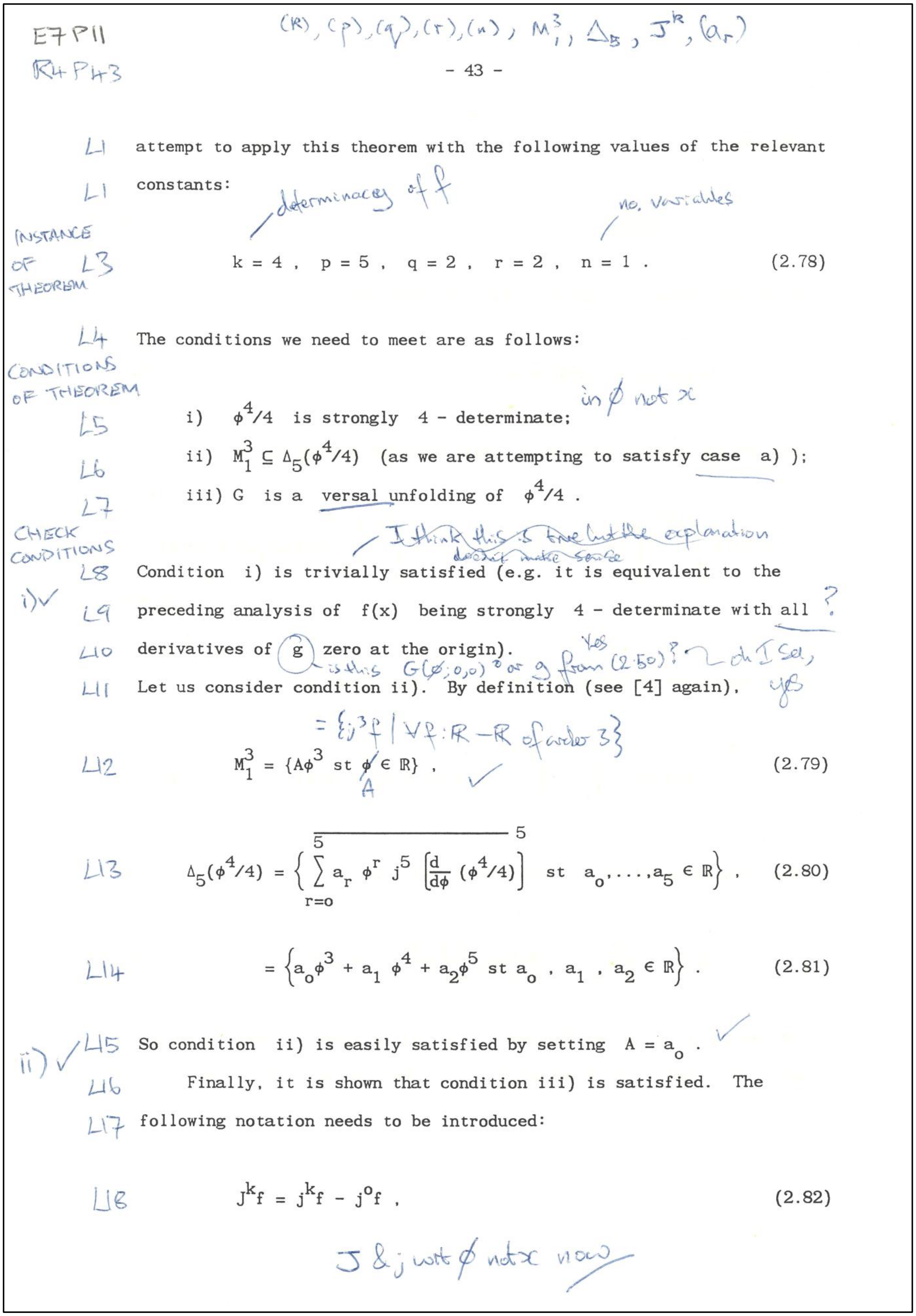
L10
$$x(\phi) = \left[\frac{3}{g(o)}\right]^{1/4}\phi - \frac{g'(o)}{20g(o)}\left[\frac{3}{g(o)}\right]^{1/2}\phi^2 + O(\phi^3) \ . \qquad (2.77)$$

END OF COROLLARY

p153

L11 The next step is to use theorem 8.7 in [4] to prove the existence of a
L12 more simple unfolding from which G may be induced. We therefore

STEP 2
G → H



E7P11 $(k), (p), (q), (r), (n), M^3_1, \Delta_5, J^k, (a_r)$

R4P43

- 43 -

L1 attempt to apply this theorem with the following values of the relevant

L1 constants: determinacy of f no. variables

INSTANCE OF THEOREM L3

$$k = 4\ ,\quad p = 5\ ,\quad q = 2\ ,\quad r = 2\ ,\quad n = 1\ . \qquad (2.78)$$

L4 The conditions we need to meet are as follows:

CONDITIONS OF THEOREM

L5 i) $\phi^4/4$ is strongly 4 - determinate; in ϕ not x

L6 ii) $M^3_1 \subseteq \Delta_5(\phi^4/4)$ (as we are attempting to satisfy case a));

L7 iii) G is a versal unfolding of $\phi^4/4$.

CHECK CONDITIONS

I think this is true but the explanation doesn't make sense

L8 Condition i) is trivially satisfied (e.g. it is equivalent to the

i) ✓ L9 preceding analysis of f(x) being strongly 4 - determinate with all ?

L10 derivatives of g zero at the origin). is this $G(\phi;0,0)$? or g from (2.50)? Yes — oh I see, yes

L11 Let us consider condition ii). By definition (see [4] again),

$$= \{j^3 f \mid \forall f: \mathbb{R} - \mathbb{R} \text{ of order } 3\}$$

L12
$$M^3_1 = \{A\phi^3 \text{ st } \overset{A}{\cancel{\phi}} \in \mathbb{R}\}\ , \qquad (2.79)$$
✓

L13
$$\Delta_5(\phi^4/4) = \left\{ \overline{\sum_{r=o}^{5} a_r\ \phi^r\ j^5 \left[\frac{d}{d\phi}(\phi^4/4)\right]}^{\,5} \text{ st } a_o,\dots,a_5 \in \mathbb{R}\right\}\ , \qquad (2.80)$$

L14
$$= \left\{a_o\phi^3 + a_1\ \phi^4 + a_2\phi^5 \text{ st } a_o\ ,\ a_1\ ,\ a_2 \in \mathbb{R}\right\}\ . \qquad (2.81)$$

ii) ✓ L15 So condition ii) is easily satisfied by setting $A = a_o$. ✓

L16 Finally, it is shown that condition iii) is satisfied. The

L17 following notation needs to be introduced:

L18
$$J^k f = j^k f - j^o f\ , \qquad (2.82)$$

J & j wrt ϕ not x now

E7 P12

R4 P44

$v_a^k(G)$, $v_b^k(G)$, J_1^k, $V^k(G)$, span

L1
$$v_a^k(G) = \frac{\partial}{\partial a}\left[J^k(G(\phi;a,\underset{0}{\cancel{0}}))\right] , \tag{2.83}$$

derivatives
treat G as a fn of a & b

L2 and similarly,

L3
$$v_b^k(G) = \frac{\partial}{\partial b}\left[J^k(G(\phi;\overset{0}{\cancel{0}},b))\right] . \tag{2.84}$$

L4 Let $V^k(G) = \text{span}\ (v_a^k(G),\ v_b^k(G))$. (2.85)

the set of functions of ϕ spanned by $v_a^k(G)$ & $v_b^k(G)$

P147

L5 Now, theorem 8.6 in [4] states that G is versal when $V^k(G)$ and

L6 $\Delta_k(\phi^4/4)$ are transverse subspaces of J_1^k , where $\frac{\phi^4}{4}$ is k-determinate.

L7 $\left[J_1^k = \left\{\sum_{r=1}^{k} a_r\phi^r \text{ st } a_1,\ldots,a_k \in \mathbb{R}\right\}\right]$. Clearly, here, $k = 4$. Equations

L8 (2.76) and (2.83) with $k = 4$ give

L9
$$v_a^4(G) = J^4\int_o^{x(\phi)}\left[\int_o^{y} h(z)dz\right]dy \tag{2.86}$$

L10
$$v_b^4(G) = J^4x(\phi) . \tag{2.87}$$

L11 We have the general result for $u = u(\phi)$ that

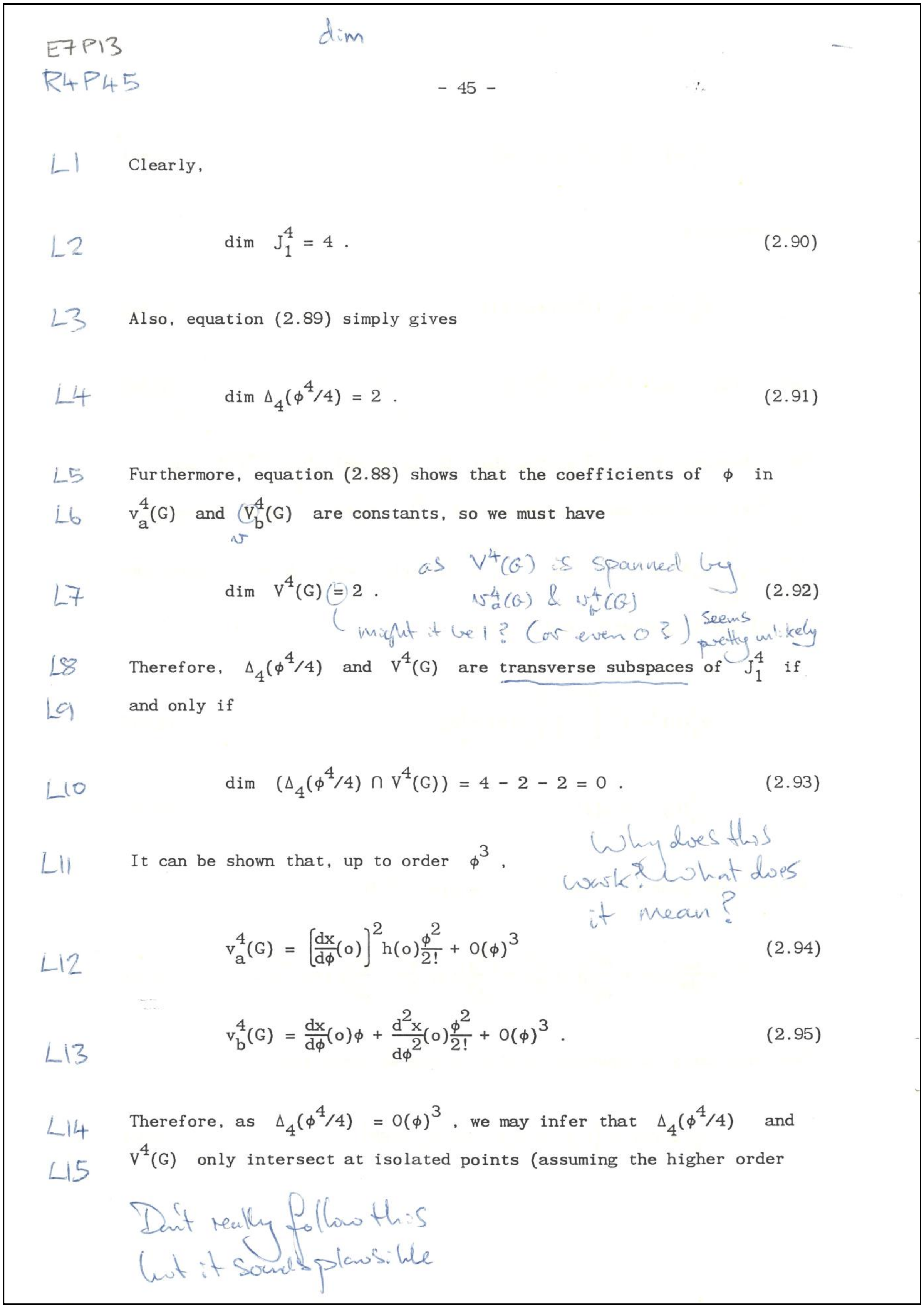
L12
$$J^4u = \frac{du}{d\phi}(o)\phi + \frac{d^2u}{d\phi^2}(o)\ \frac{\phi^2}{2!} + \frac{d^3u}{d\phi^3}\ (o)\ \frac{\phi^3}{3!} + \frac{d^4u}{d\phi^4}\ (o)\ \frac{\phi^4}{4!} . \tag{2.88}$$

✓

L13 Now, similarly to equation (2.81), it can be shown that

L14
$$\Delta_{\textcircled{k}}(\phi^4/4) = \left\{a_o\phi^3 + a_1\phi^4 \text{ st } a_o,a_1\in\mathbb{R}\right\} . \tag{2.89}$$

where k = 4



E7P13
R4P45

dim

- 45 -

L1 Clearly,

L2
$$\dim \ J_1^4 = 4 \ . \qquad (2.90)$$

L3 Also, equation (2.89) simply gives

L4
$$\dim \Delta_4(\phi^4/4) = 2 \ . \qquad (2.91)$$

L5 Furthermore, equation (2.88) shows that the coefficients of ϕ in

L6 $v_a^4(G)$ and $\overset{}{\underset{v}{\left(V_b^4\right.}}(G)$ are constants, so we must have

L7
$$\dim \ V^4(G) \boxed{=} 2 \ . \qquad (2.92)$$

as $V^4(G)$ is spanned by $v_a^4(G)$ & $v_b^4(G)$

(might it be 1? (or even 0?) Seems pretty unlikely

L8 Therefore, $\Delta_4(\phi^4/4)$ and $V^4(G)$ are transverse subspaces of J_1^4 if

L9 and only if

L10
$$\dim \ (\Delta_4(\phi^4/4) \cap V^4(G)) = 4 - 2 - 2 = 0 \ . \qquad (2.93)$$

L11 It can be shown that, up to order ϕ^3,

Why does this work? What does it mean?

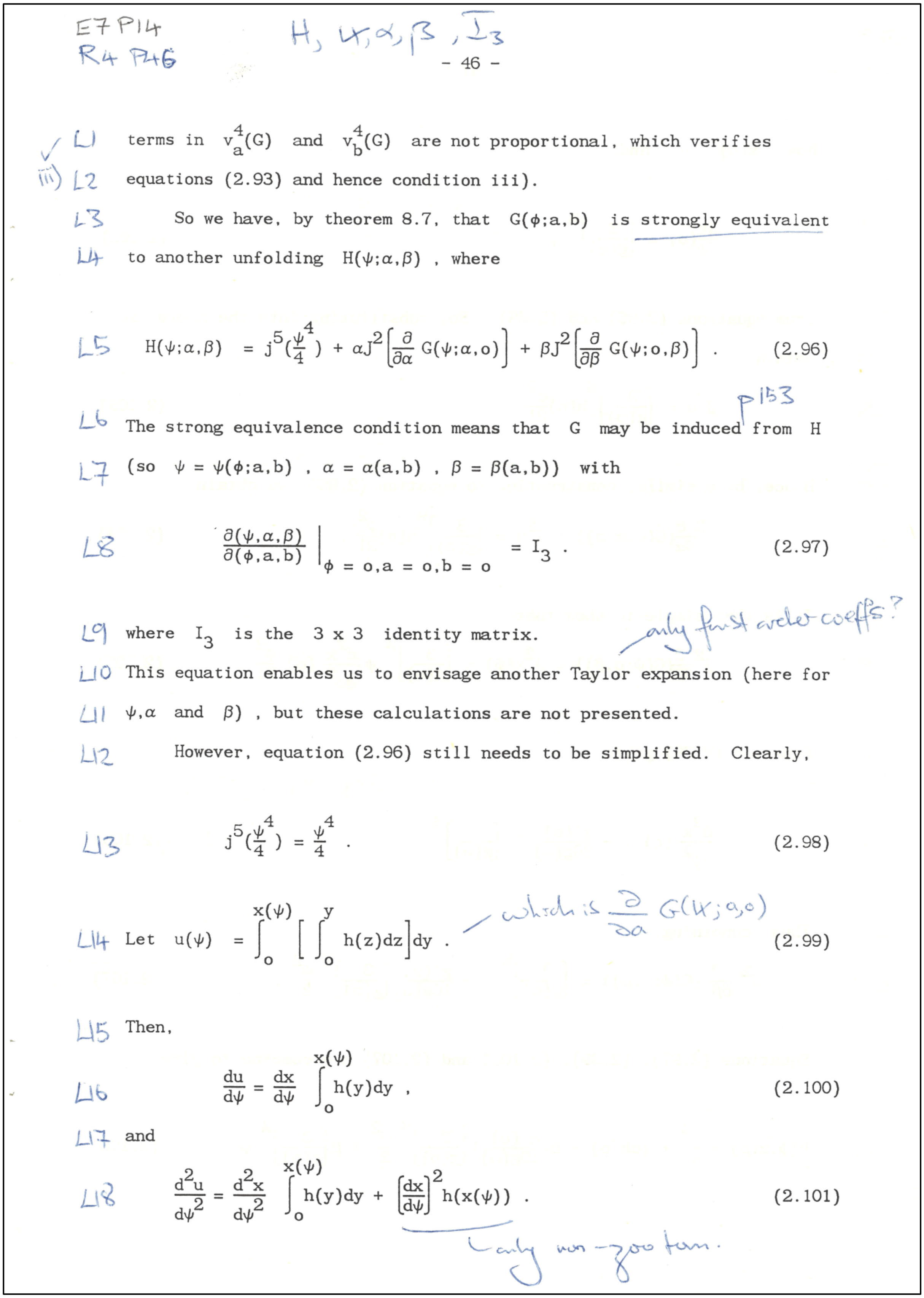
L12
$$v_a^4(G) = \left[\frac{dx}{d\phi}(o)\right]^2 h(o)\frac{\phi^2}{2!} + 0(\phi)^3 \qquad (2.94)$$

L13
$$v_b^4(G) = \frac{dx}{d\phi}(o)\phi + \frac{d^2x}{d\phi^2}(o)\frac{\phi^2}{2!} + 0(\phi)^3 \ . \qquad (2.95)$$

L14 Therefore, as $\Delta_4(\phi^4/4) = 0(\phi)^3$, we may infer that $\Delta_4(\phi^4/4)$ and

L15 $V^4(G)$ only intersect at isolated points (assuming the higher order

Don't really follow this
but it sounds plausible



E7 P14
R4 P46

$H, \psi, \alpha, \beta, I_3$

✓ iii) L1 terms in $v_a^4(G)$ and $v_b^4(G)$ are not proportional, which verifies
L2 equations (2.93) and hence condition iii).

L3 So we have, by theorem 8.7, that $G(\phi;a,b)$ is strongly equivalent
L4 to another unfolding $H(\psi;\alpha,\beta)$, where

L5
$$H(\psi;\alpha,\beta) = j^5\left(\frac{\psi^4}{4}\right) + \alpha J^2\left[\frac{\partial}{\partial\alpha} G(\psi;\alpha,o)\right] + \beta J^2\left[\frac{\partial}{\partial\beta} G(\psi;o,\beta)\right] . \qquad (2.96)$$

L6 The strong equivalence condition means that G may be induced p153 from H
L7 (so $\psi = \psi(\phi;a,b)$, $\alpha = \alpha(a,b)$, $\beta = \beta(a,b)$) with

L8
$$\left.\frac{\partial(\psi,\alpha,\beta)}{\partial(\phi,a,b)}\right|_{\phi = o, a = o, b = o} = I_3 . \qquad (2.97)$$

L9 where I_3 is the 3 x 3 identity matrix. only first order coeffs?
L10 This equation enables us to envisage another Taylor expansion (here for
L11 ψ,α and β), but these calculations are not presented.

L12 However, equation (2.96) still needs to be simplified. Clearly,

L13
$$j^5\left(\frac{\psi^4}{4}\right) = \frac{\psi^4}{4} . \qquad (2.98)$$

L14 Let
$$u(\psi) = \int_o^{x(\psi)} \left[\int_o^y h(z)dz \right] dy . \qquad (2.99)$$
which is $\frac{\partial}{\partial a} G(\psi;a,o)$

L15 Then,

L16
$$\frac{du}{d\psi} = \frac{dx}{d\psi} \int_o^{x(\psi)} h(y)dy , \qquad (2.100)$$

L17 and

L18
$$\frac{d^2u}{d\psi^2} = \frac{d^2x}{d\psi^2} \int_o^{x(\psi)} h(y)dy + \underbrace{\left[\frac{dx}{d\psi}\right]^2 h(x(\psi))}_{\text{only non-zero term.}} . \qquad (2.101)$$

E7P15
R4P47

L1 Now, $x(o) = o$, and

L2
$$\frac{dx}{d\psi}(o) = \left[\frac{3}{g(o)}\right]^{1/4} , \qquad (2.102)$$

(2.77) better

L3 from equations (2.66) and (2.68). So, substituting into the above, we
L4 obtain

L5
$$J^2 u = \left[\frac{3}{g(o)}\right]^{1/2} h(o)\frac{\psi^2}{2!} . \qquad (2.103)$$

? this is trivial

L6 Hence, by a similar construction to equation (2.86), we obtain

L7
$$J^2\frac{\partial}{\partial\alpha}(G(\psi;\alpha,o)) = J^2 u = \left[\frac{3}{g(o)}\right]^{1/2} h(o)\frac{\psi^2}{2\textcircled{!}} . \qquad (2.104)$$

no need

L8 It is then simple to show that

L9
$$J^2\frac{\partial}{\partial\beta}(G(\psi;o,\beta)) = J^2 x(\psi) = \left[\frac{3}{g(o)}\right]^{1/4} \psi \textcircled{+} \frac{d^2x}{d\psi^2}(o)\ \frac{\psi^2}{2!} . \qquad (2.105)$$

L10 Equation (2.78) gives

L11
$$\frac{d^2x}{d\psi^2}(o) = -\frac{g'(o)}{20g(o)}\ 2\left[\frac{3}{g(o)}\right]^{1/2} . \qquad (2.106)$$

Again, Simply use (2.77)

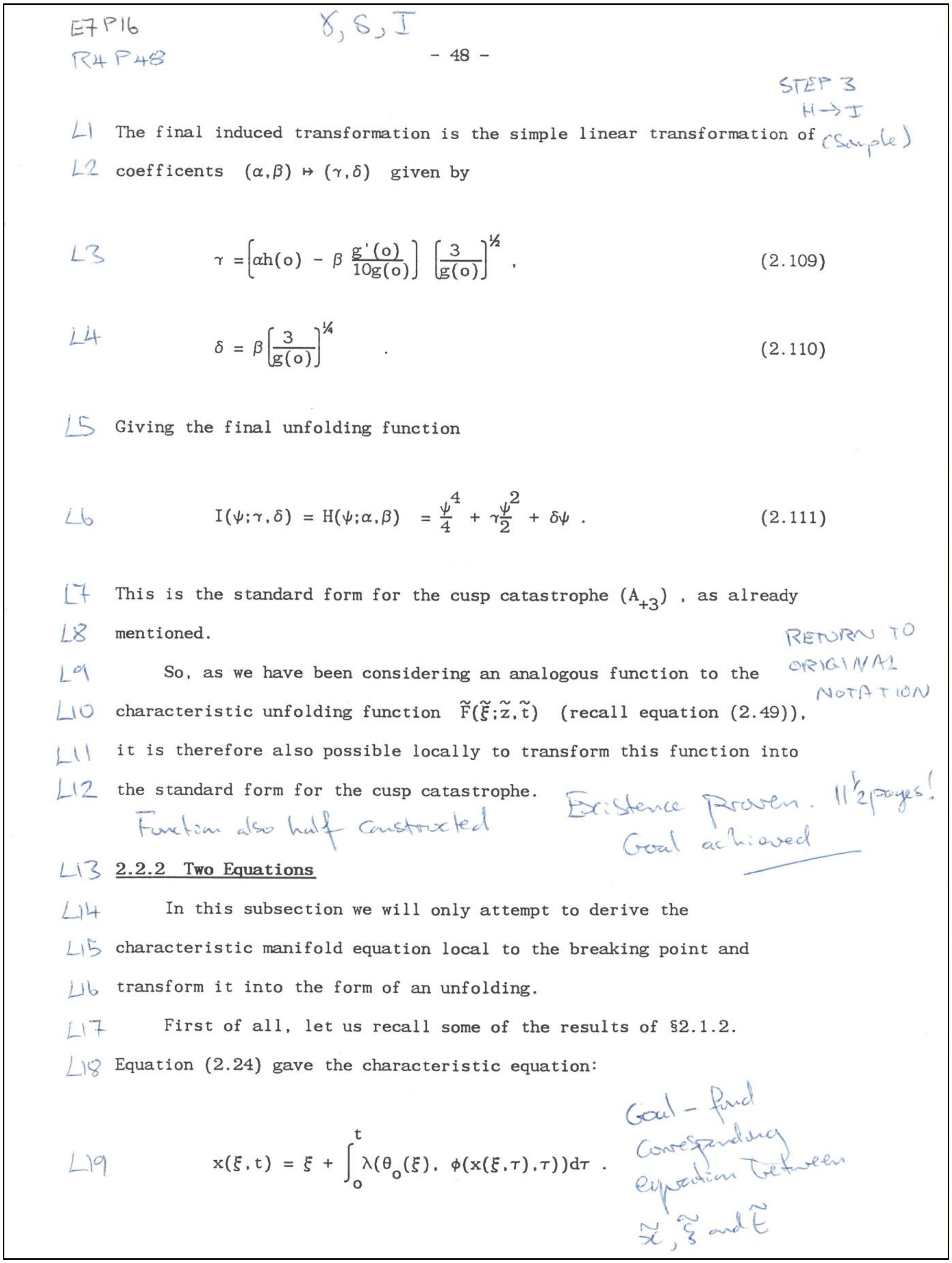
L12 Thus, combining,

L13
$$J^2\ \frac{\partial}{\partial\beta}\ (G(\psi;o,\beta)) = \left[\frac{3}{g(o)}\right]^{1/4} \psi - \frac{g'(o)}{10g(o)}\left[\frac{3}{g(o)}\right]^{1/2} \frac{\psi^2}{2} . \qquad (2.107)$$

L14 Equations (2.97), (2.99), (2.105) and (2.107) now combine to give

~~L15~~
L15
$$H(\psi;\alpha,\beta) = \frac{\psi^4}{4} + \{\alpha h(o) - \beta\,\frac{g'(o)}{10g(o)}\}\left[\frac{3}{g(o)}\right]^{1/2}\frac{\psi^2}{2} + \beta\left[\frac{3}{g(o)}\right]^{1/4}\psi . \qquad (2.108)$$

END OF
STEP 2



E7 P16
R4 P48

γ, δ, I

- 48 -

STEP 3
$H \to I$
(Simple)

The final induced transformation is the simple linear transformation of coefficents $(\alpha,\beta) \mapsto (\gamma,\delta)$ given by

$$\gamma = \left[\alpha h(o) - \beta \frac{g'(o)}{10g(o)}\right] \left[\frac{3}{g(o)}\right]^{½} , \qquad (2.109)$$

$$\delta = \beta \left[\frac{3}{g(o)}\right]^{¼} . \qquad (2.110)$$

Giving the final unfolding function

$$I(\psi;\gamma,\delta) = H(\psi;\alpha,\beta) = \frac{\psi^4}{4} + \gamma\frac{\psi^2}{2} + \delta\psi . \qquad (2.111)$$

This is the standard form for the cusp catastrophe (A_{+3}) , as already mentioned.

RETURN TO ORIGINAL NOTATION

So, as we have been considering an analogous function to the characteristic unfolding function $\tilde{F}(\tilde{\xi};\tilde{z},\tilde{t})$ (recall equation (2.49)), it is therefore also possible locally to transform this function into the standard form for the cusp catastrophe.

Existence proven. 1½ pages!
Goal achieved

Function also half constructed

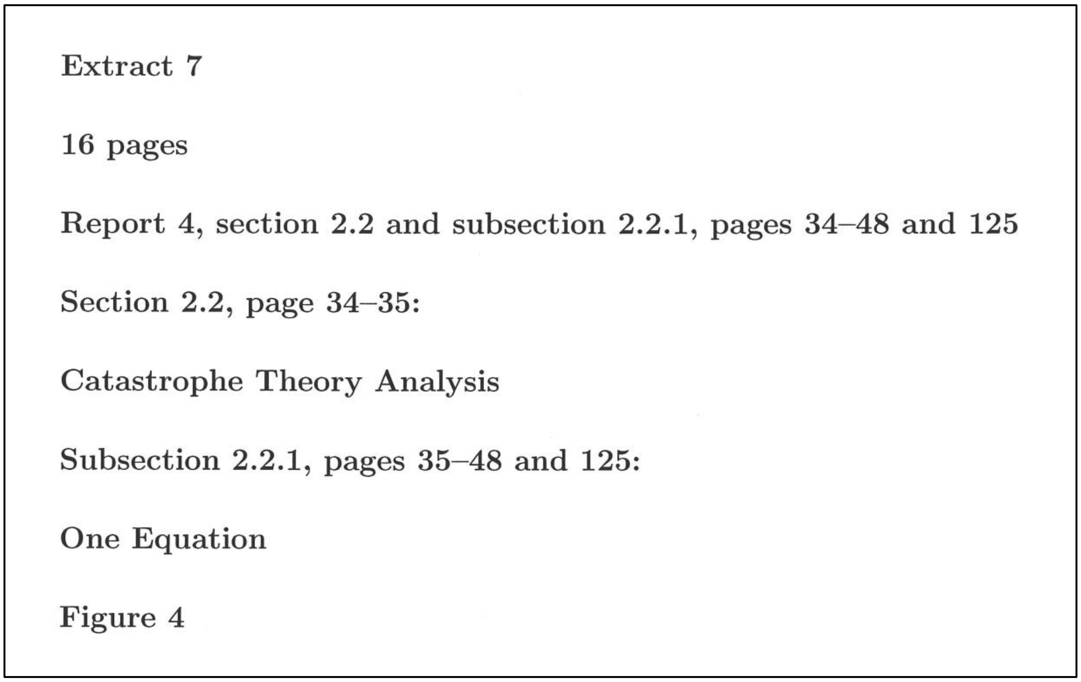
2.2.2 Two Equations

In this subsection we will only attempt to derive the characteristic manifold equation local to the breaking point and transform it into the form of an unfolding.

First of all, let us recall some of the results of §2.1.2.
Equation (2.24) gave the characteristic equation:

$$x(\xi,t) = \xi + \int_o^t \lambda(\theta_o(\xi),\ \phi(x(\xi,\tau),\tau))d\tau .$$

Goal – find corresponding equation between $\tilde{x}$, $\tilde{\xi}$ and $\tilde{t}$



Extract 7

16 pages

Report 4, section 2.2 and subsection 2.2.1, pages 34–48 and 125

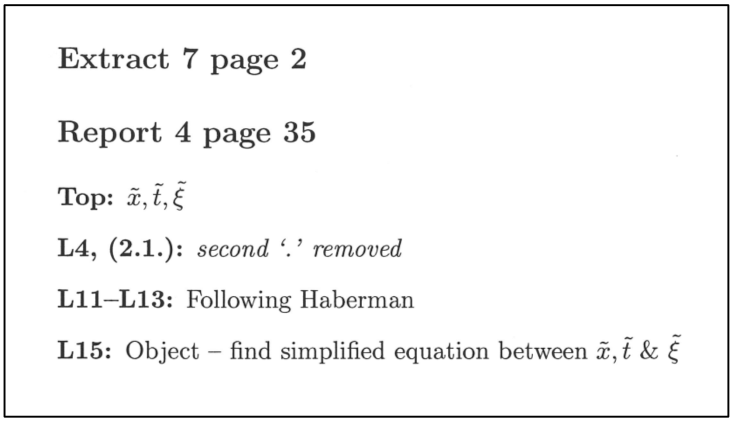
Section 2.2, page 34–35:

Catastrophe Theory Analysis

Subsection 2.2.1, pages 35–48 and 125:

One Equation

Figure 4



Extract 7 page 2

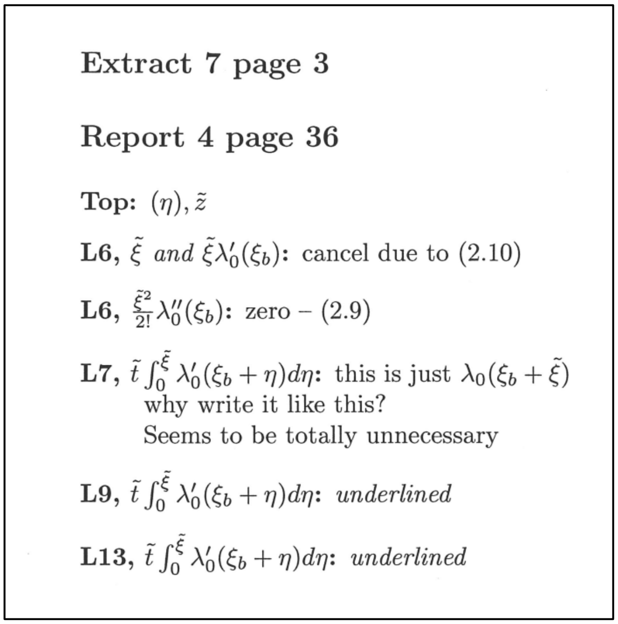
Report 4 page 35

Top: $\tilde{x}, \tilde{t}, \tilde{\xi}$

L4, (2.1.): *second '.' removed*

L11–L13: Following Haberman

L15: Object – find simplified equation between $\tilde{x}, \tilde{t}$ & $\tilde{\xi}$



Extract 7 page 3

Report 4 page 36

Top: $(\eta), \tilde{z}$

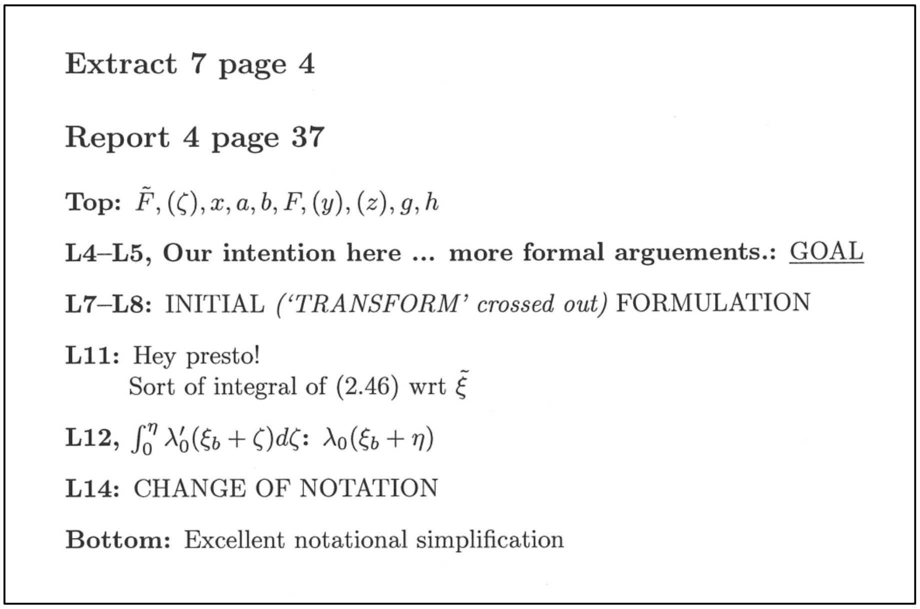
L6, $\tilde{\xi}$ *and* $\tilde{\xi}\lambda_0'(\xi_b)$**:** cancel due to (2.10)

L6, $\frac{\tilde{\xi}^2}{2!}\lambda_0''(\xi_b)$**:** zero – (2.9)

L7, $\tilde{t}\int_0^{\tilde{\xi}} \lambda_0'(\xi_b+\eta)d\eta$**:** this is just $\lambda_0(\xi_b+\tilde{\xi})$
why write it like this?
Seems to be totally unnecessary

L9, $\tilde{t}\int_0^{\tilde{\xi}} \lambda_0'(\xi_b+\eta)d\eta$**:** *underlined*

L13, $\tilde{t}\int_0^{\tilde{\xi}} \lambda_0'(\xi_b+\eta)d\eta$**:** *underlined*



Extract 7 page 4

Report 4 page 37

Top: $\tilde{F}, (\zeta), x, a, b, F, (y), (z), g, h$

L4–L5, Our intention here ... more formal arguements.: GOAL

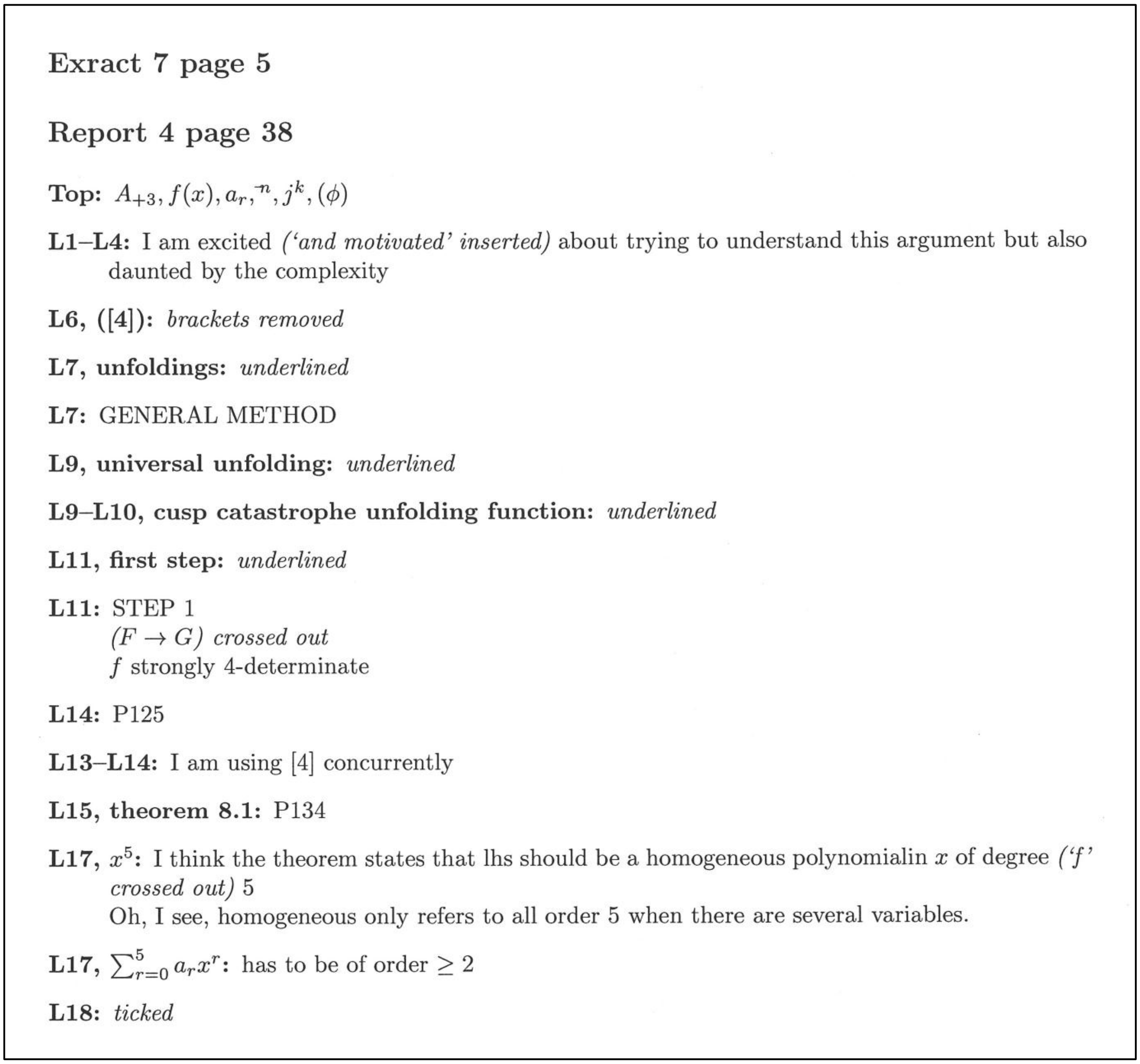
L7–L8: INITIAL *('TRANSFORM' crossed out)* FORMULATION

L11: Hey presto!
Sort of integral of (2.46) wrt $\tilde{\xi}$

L12, $\int_0^{\eta} \lambda_0'(\xi_b+\zeta)d\zeta$**:** $\lambda_0(\xi_b+\eta)$

L14: CHANGE OF NOTATION

Bottom: Excellent notational simplification



Exract 7 page 5

Report 4 page 38

Top: $A_{+3}, f(x), a_r, ^{-n}, j^k, (\phi)$

L1–L4: I am excited *(‘and motivated’ inserted)* about trying to understand this argument but also daunted by the complexity

L6, ([4]): *brackets removed*

L7, unfoldings: *underlined*

L7: GENERAL METHOD

L9, universal unfolding: *underlined*

L9–L10, cusp catastrophe unfolding function: *underlined*

L11, first step: *underlined*

L11: STEP 1
($F \to G$) crossed out
f strongly 4-determinate

L14: P125

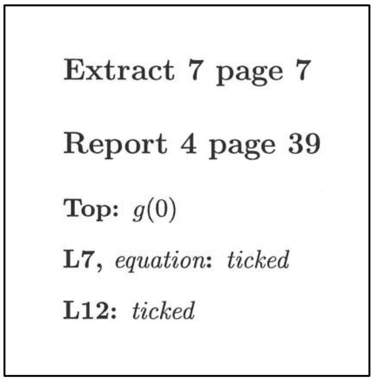
L13–L14: I am using [4] concurrently

L15, theorem 8.1: P134

L17, x^5**:** I think the theorem states that lhs should be a homogeneous polynomialin x of degree *(‘f’ crossed out)* 5
Oh, I see, homogeneous only refers to all order 5 when there are several variables.

L17, $\sum_{r=0}^{5} a_r x^r$**:** has to be of order ≥ 2

L18: *ticked*



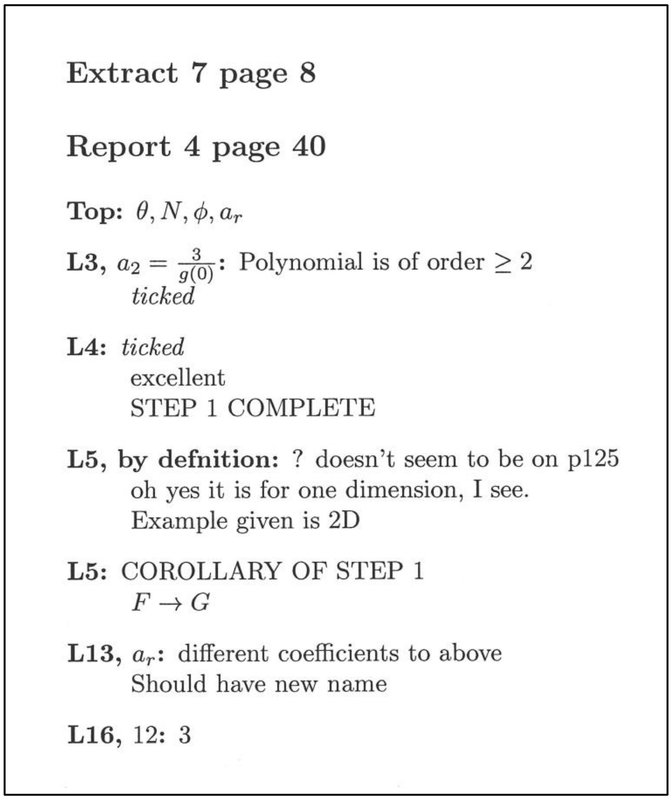
Extract 7 page 7

Report 4 page 39

Top: $g(0)$

L7, *equation***:** *ticked*

L12: *ticked*



Extract 7 page 8

Report 4 page 40

Top: θ, N, ϕ, a_r

L3, $a_2 = \frac{3}{g(0)}$**:** Polynomial is of order ≥ 2
ticked

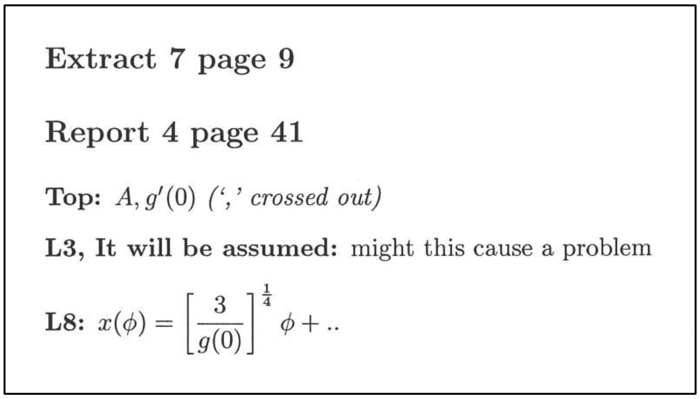
L4: *ticked*
excellent
STEP 1 COMPLETE

L5, by defnition: ? doesn't seem to be on p125
oh yes it is for one dimension, I see.
Example given is 2D

L5: COROLLARY OF STEP 1
$F \to G$

L13, a_r**:** different coefficients to above
Should have new name

L16, 12**:** 3



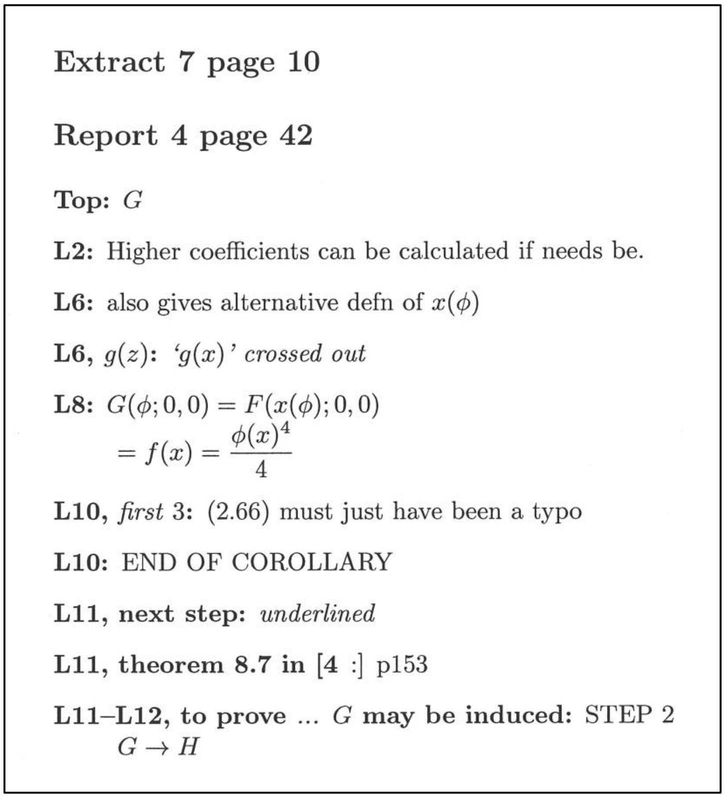
Extract 7 page 9

Report 4 page 41

Top: $A, g'(0)$ *(‘,’ crossed out)*

L3, It will be assumed: might this cause a problem

L8: $x(\phi) = \left[\frac{3}{g(0)}\right]^{\frac{1}{4}} \phi + ..$



Extract 7 page 10

Report 4 page 42

Top: G

L2: Higher coefficients can be calculated if needs be.

L6: also gives alternative defn of $x(\phi)$

L6, $g(z)$**:** '$g(x)$' *crossed out*

L8: $G(\phi; 0, 0) = F(x(\phi); 0, 0)$

$$= f(x) = \frac{\phi(x)^4}{4}$$

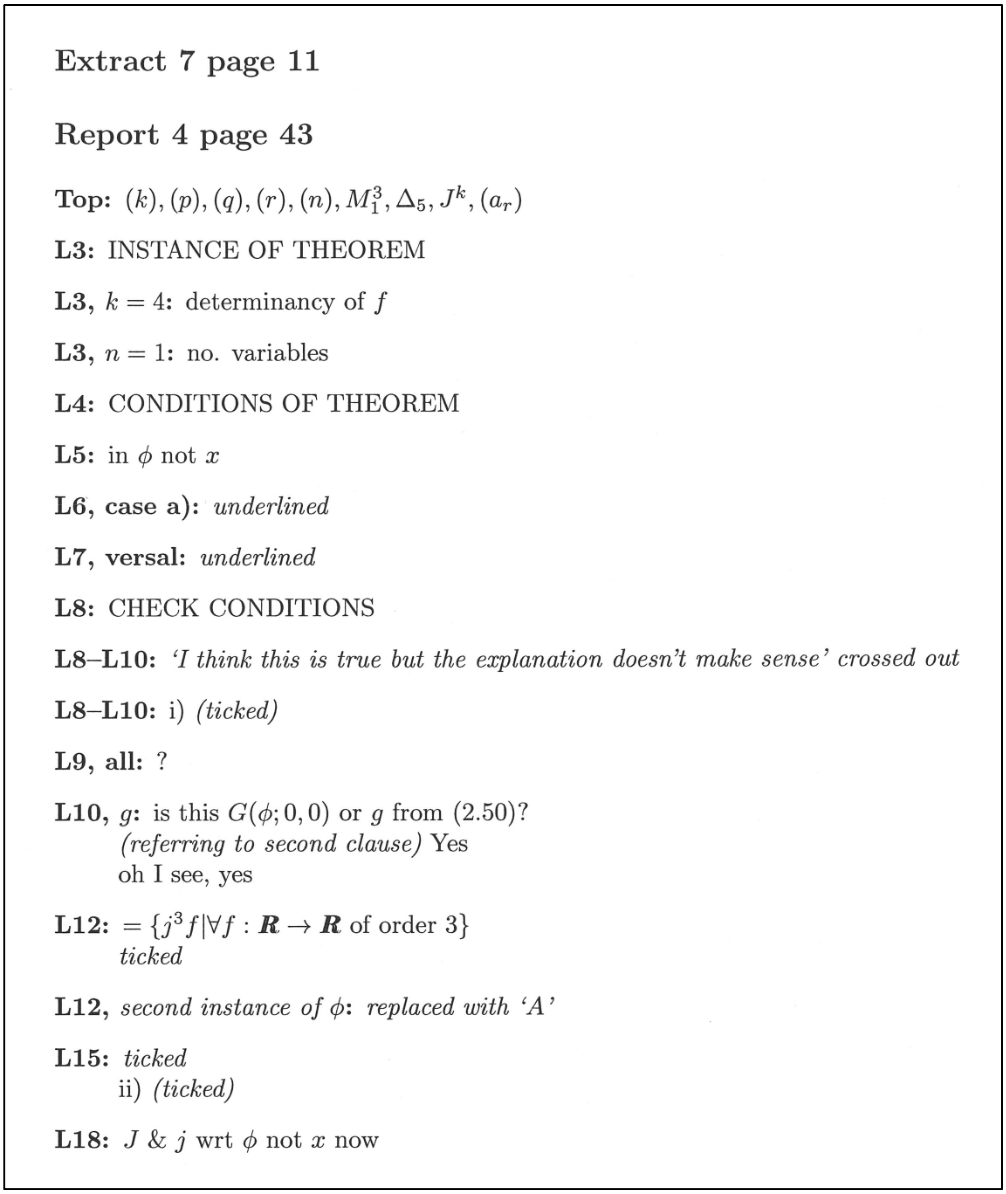
L10, *first* 3**:** (2.66) must just have been a typo

L10: END OF COROLLARY

L11, next step: *underlined*

L11, theorem 8.7 in [4 :] p153

L11–L12, to prove ... G **may be induced:** STEP 2 $G \to H$



Extract 7 page 11

Report 4 page 43

Top: $(k), (p), (q), (r), (n), M_1^3, \Delta_5, J^k, (a_r)$

L3: INSTANCE OF THEOREM

L3, $k = 4$: determinancy of f

L3, $n = 1$: no. variables

L4: CONDITIONS OF THEOREM

L5: in ϕ not x

L6, case a): *underlined*

L7, versal: *underlined*

L8: CHECK CONDITIONS

L8–L10: *'I think this is true but the explanation doesn't make sense' crossed out*

L8–L10: i) *(ticked)*

L9, all: ?

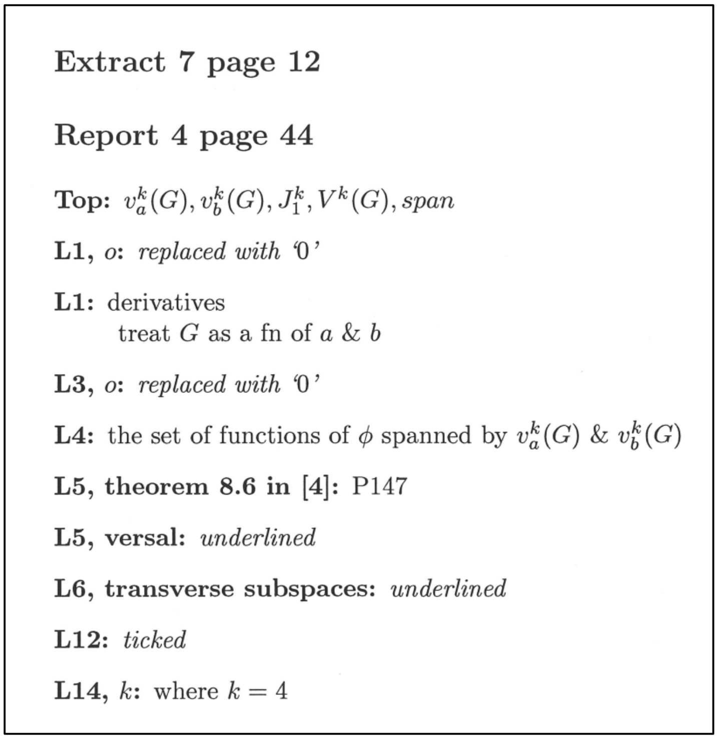
L10, g: is this $G(\phi; 0, 0)$ or g from (2.50)?
(referring to second clause) Yes
oh I see, yes

L12: $= \{j^3 f | \forall f : \boldsymbol{R} \to \boldsymbol{R}$ of order $3\}$
ticked

L12, *second instance of* ϕ**:** *replaced with 'A'*

L15: *ticked*
ii) *(ticked)*

L18: J & j wrt ϕ not x now



Extract 7 page 12

Report 4 page 44

Top: $v_a^k(G), v_b^k(G), J_1^k, V^k(G), span$

L1, o: *replaced with '0'*

L1: derivatives
treat G as a fn of a & b

L3, o: *replaced with '0'*

L4: the set of functions of ϕ spanned by $v_a^k(G)$ & $v_b^k(G)$

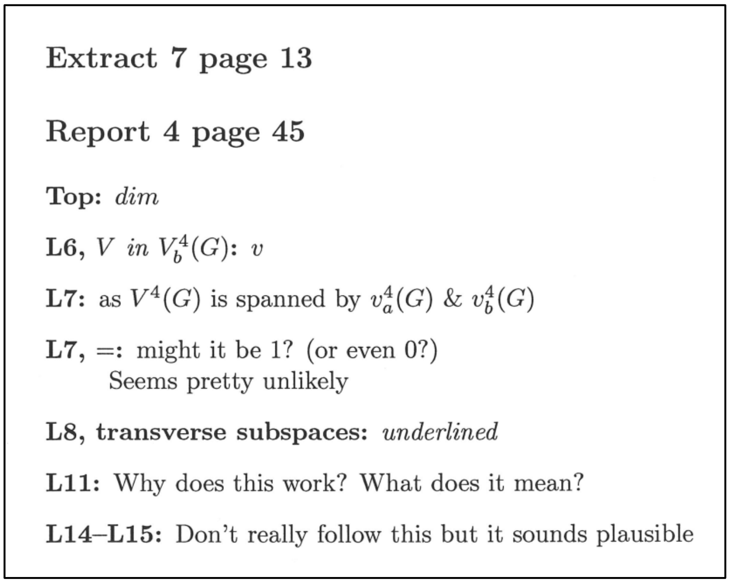
L5, theorem 8.6 in [4]: P147

L5, versal: *underlined*

L6, transverse subspaces: *underlined*

L12: *ticked*

L14, k: where $k = 4$



Extract 7 page 13

Report 4 page 45

Top: *dim*

L6, V *in* $V_b^4(G)$**:** v

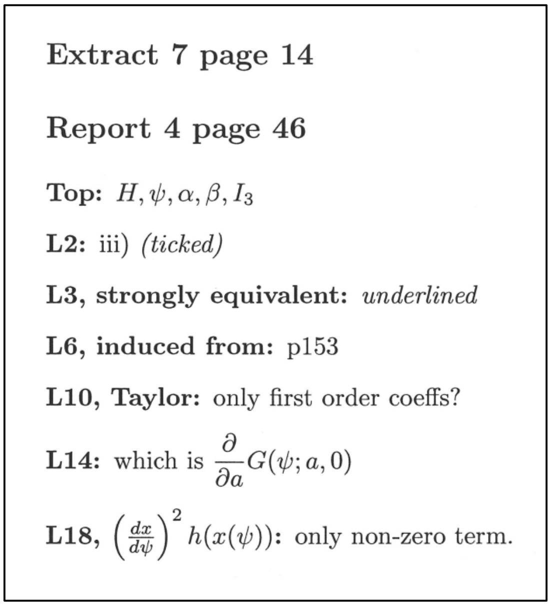
L7: as $V^4(G)$ is spanned by $v_a^4(G)$ & $v_b^4(G)$

L7, =: might it be 1? (or even 0?)
Seems pretty unlikely

L8, transverse subspaces: *underlined*

L11: Why does this work? What does it mean?

L14–L15: Don't really follow this but it sounds plausible



Extract 7 page 14

Report 4 page 46

Top: $H, \psi, \alpha, \beta, I_3$

L2: iii) *(ticked)*

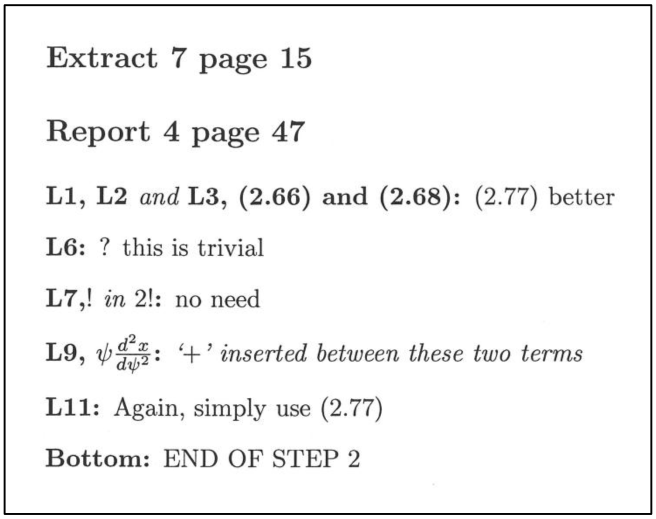
L3, strongly equivalent: *underlined*

L6, induced from: p153

L10, Taylor: only first order coeffs?

L14: which is $\dfrac{\partial}{\partial a} G(\psi; a, 0)$

L18, $\left(\frac{dx}{d\psi}\right)^2 h(x(\psi))$**:** only non-zero term.



Extract 7 page 15

Report 4 page 47

L1, L2 *and* **L3, (2.66) and (2.68):** (2.77) better

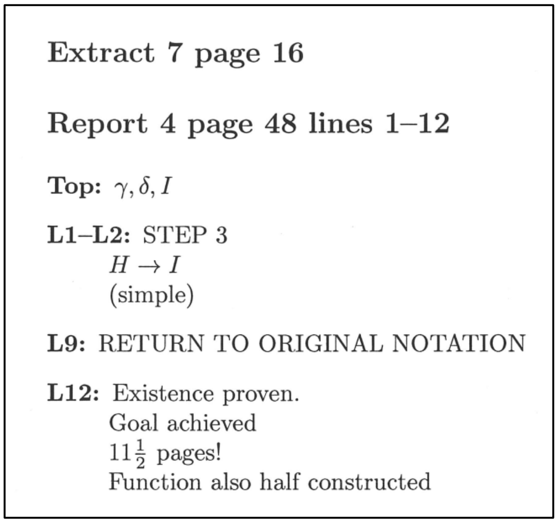
L6: ? this is trivial

L7,! *in* 2!**:** no need

L9, $\psi \frac{d^2x}{d\psi^2}$**:** *'+' inserted between these two terms*

L11: Again, simply use (2.77)

Bottom: END OF STEP 2



Extract 7 page 16

Report 4 page 48 lines 1–12

Top: γ, δ, I

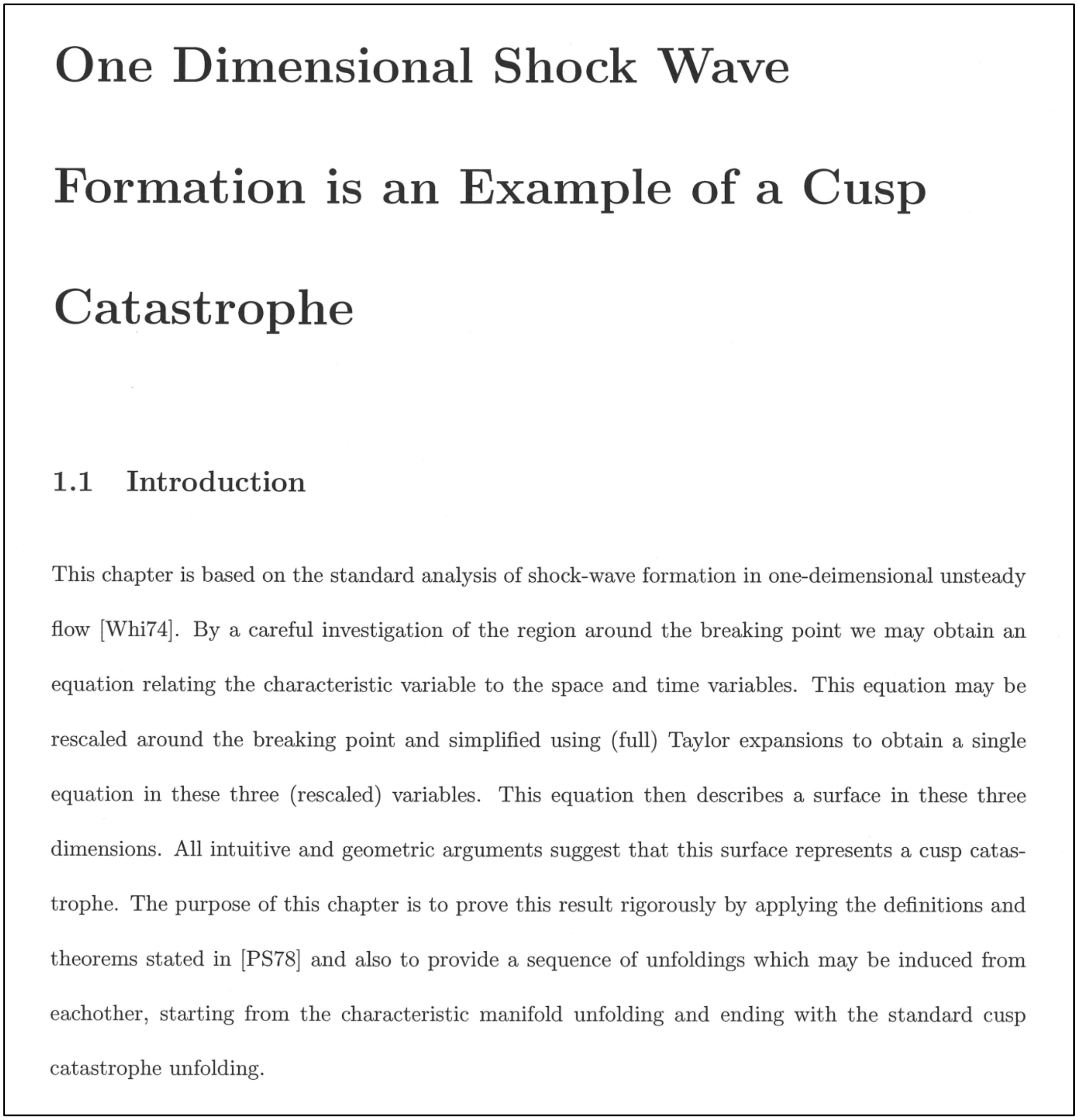
L1–L2: STEP 3
$H \to I$
(simple)

L9: RETURN TO ORIGINAL NOTATION

L12: Existence proven.
Goal achieved
$11\frac{1}{2}$ pages!
Function also half constructed

Appendix E

Entire Final Proof



One Dimensional Shock Wave Formation is an Example of a Cusp Catastrophe

1.1 Introduction

This chapter is based on the standard analysis of shock-wave formation in one-deimensional unsteady flow [Whi74]. By a careful investigation of the region around the breaking point we may obtain an equation relating the characteristic variable to the space and time variables. This equation may be rescaled around the breaking point and simplified using (full) Taylor expansions to obtain a single equation in these three (rescaled) variables. This equation then describes a surface in these three dimensions. All intuitive and geometric arguments suggest that this surface represents a cusp catastrophe. The purpose of this chapter is to prove this result rigorously by applying the definitions and theorems stated in [PS78] and also to provide a sequence of unfoldings which may be induced from eachother, starting from the characteristic manifold unfolding and ending with the standard cusp catastrophe unfolding.

Recently, the formation and propagation of shock waves for a single conservation law in multiple space dimensions and its connection with all possible geometric singularities has been fully ivestiagted by Izumiya and Kossioris [IK97]. The formation of shocks as an example of a cusp catastrophe in mulitple space dimensions was originally studied by Nakane [Nak88]. This chapter, however, provides a specific proof for the case of shock wave formation in one space dimension. This case was previously investigated by Guckenheimer [Guc75].

1.2 Derivation of the Rescaled Caustic Equation

In this section we shall derive the rescaled characteristic equation from first principles.

We recall the following formulae from volume I:

Let us consider a single conservation law (also called the Cauchy problem):

$$\left.\begin{array}{lrcl} \text{(I:6.1)} & \dfrac{\partial\psi}{\partial t} + c(\psi)\dfrac{\partial\psi}{\partial x} & = & 0 \\ \text{(I:8.3)} & \psi(x,0) & = & \psi_0(x) \\ \text{(I:8.4)} & c(\psi(x,0)) & = & c_0(x) \end{array}\right\} \tag{1.1}$$

As in [Sch73] let us assume that

$$\psi_0, c \in C^\infty[\boldsymbol{R} \to \boldsymbol{R}] \tag{1.2}$$

and $c(\psi)$ is uniformly convex, ie.

$$\exists\varepsilon > 0 \text{ such that } \forall\psi \in \boldsymbol{R}, \frac{d^2c}{d\psi^2} \geq \varepsilon \tag{1.3}$$

Then it is well known, eg. [Lax54, Whi74], that initially smooth solutions are found by using the method of characteristics. Let us introduce a family of characteristic curves F_Γ, as in volume I:

$$\left.\begin{array}{lrcl} \text{(I:6.14)} & F_\Gamma & = & \{\Gamma(\xi) \mid \xi \in \boldsymbol{R}\} \\ \text{(I:8.5)} & \Gamma(\xi): \quad x & = & \xi + c_0(\xi)t \end{array}\right\} \tag{1.4}$$

Then it was shown in (I:§6.1) that (1.1) implies that ψ is constant on $\Gamma(\xi)$. Following [Whi74], let us consider an isolated inflection point in c_0 given by the characteristic curve $\Gamma(\xi_B)$:

$$\left.\begin{array}{rrcl} \text{(I:8.11)} & c_0^{(2)}(\xi_B) & = & 0 \\ \exists\varepsilon > 0 \text{ such that } \forall\xi \in [\xi_B - \varepsilon, \xi_B + \varepsilon] \setminus \{\xi_B\}, & c_0^{(2)}(\xi) & \neq & 0 \\ \text{where, for all functions } \phi(x), & \phi^{(n)}(x) & = & \dfrac{d^n\phi}{dx^n} \end{array}\right\} \tag{1.5}$$

Then, if we define t_B and x_B by:

$$\left.\begin{array}{rrcl} \text{(I:8.10)} & t_B & = & -\dfrac{1}{c_0^{(1)}(\xi_B)} \\ & x_B & = & \xi_B + c_0(\xi_B)t_B \end{array}\right\} \tag{1.6}$$

Then, provided the following conditions on c hold:

$$\left.\begin{array}{rrcl} & c_0^{(1)}(\xi_B) & < & 0 \\ \text{(I:8.12)} & c_0^{(3)}(\xi_B) & > & 0 \end{array}\right\} \tag{1.7}$$

We have shown in (I:§8.1.2) that the breaking point is the beginning in time of the caustic curve and that these two equations hold there. However, this feature will not be part of the physical solution unless we ensure that no other characteristic brings information to the solution before this time. This is ensured by the following condition:

$$\forall\xi \in \boldsymbol{R} \setminus [\xi_L^*, \xi_R^*], \quad \frac{x_B - \xi}{c_0(\xi)} > t_B \tag{1.8}$$

where ξ_L^* and ξ_R^* are defined in (I:8.19,I:8.20). This condition ensures the local feature about (x_B, t_B) will indeed form a shock wave irrespective of the Rankine-Hugoniot jump conditions [Ran1889, Hug1870] or the particular equation of state.

An alternative condition guaranteeing no information interferes with the shock at (x_B, t_B) is to simply ensure that no shock forms before $t = t_B$ by the condition that $-\dfrac{1}{c_0^{(1)}(\xi)}$ has a global minimum at $\xi = \xi_B$. This result has not been proved here.

This existence was initially investigated by Schaeffer [Sch73]. The existence of piecewise smooth solutions with a weaker convexity condition has been proved by Jennings [Jen79].

Let us introduce the same rescaling variables as in (I:8.35):

$$\left.\begin{aligned} \tilde{x} &= x - x_B \\ \tilde{t} &= t - t_B \\ \tilde{\xi} &= \xi - \xi_B \end{aligned}\right\} \tag{1.9}$$

Then from (I:8.5), (I:8.35) and (1.6) we may deduce:

$$\tilde{x} = \tilde{\xi} + c_0(\xi_B + \tilde{\xi}) \cdot (t_B + \tilde{t}) - c_0(\xi_B)t_B \tag{1.10}$$

Introducing Taylor expansions of $c_0(\xi)$ about ξ_B and using (I:8.10) and (I:8.11) we obtain:

$$\tilde{x} = \frac{t_B}{2}\int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 c_0^{(3)}(\xi_B + \eta)d\eta + c_0(\xi_B)\tilde{t} + \tilde{t}\int_0^{\tilde{\xi}} c_0^{(1)}(\xi_B + \eta)d\eta \tag{1.11}$$

Let us now introduce the rescaled variable $\tilde{z}$, again as in (I:8.35):

$$\tilde{z} = \tilde{x} - c_0(\xi_B)\tilde{t} \tag{1.12}$$

then substituting into (1.11) gives:

$$\tilde{z} = \frac{t_B}{2}\int_0^{\tilde{\xi}} (\tilde{\xi} - \eta)^2 c_0^{(3)}(\xi_B + \eta)d\eta + \tilde{t}\int_0^{\tilde{\xi}} c_0^{(1)}(\xi_B + \eta)d\eta \tag{1.13}$$

1.3 Derivation of the Unfolding Function

In this section we shall rewrite this equation in the form of an unfolding function by integrating it.

We seek a function $\tilde{F}(\tilde{\xi}; \tilde{z}, \tilde{t})$ with the following properties:

$$\left.\begin{aligned} \forall \tilde{z}, \tilde{t} \qquad & \tilde{F}(0; \tilde{z}, \tilde{t}) &= 0 \\ (1.13) \Leftrightarrow \quad & \frac{\partial \tilde{F}}{\partial \tilde{\xi}}(\tilde{\xi}; \tilde{z}, \tilde{t}) &= 0 \end{aligned}\right\} \tag{1.14}$$

Such a function is obtained by integrating (1.13) with respect to ξ and applying suitable boundary conditions to the integrals:

$$\tilde{F}(\tilde{\xi}; \tilde{z}, \tilde{t}) = \frac{t_B}{2}\int_0^{\tilde{\xi}} \left[\int_0^{\eta} (\eta - \zeta)^2 c_0^{(3)}(\xi_B + \zeta)d\zeta\right] d\eta + \tilde{t}\int_0^{\tilde{\xi}} \left[\int_0^{\eta} c_0^{(1)}(\xi_B + \zeta)d\zeta\right] d\eta - \tilde{z}\tilde{\xi} \tag{1.15}$$

For simplicity, let us introduce the following change of notation:

$$\left.\begin{array}{rcl} g(\tilde{\xi}) & = & \frac{t_B}{2}c_0^{(3)}(\xi_B+\tilde{\xi}) \\ h(\tilde{\xi}) & = & c_0^{(1)}(\xi_B+\tilde{\xi}) \\ b(\tilde{z}) & = & -\tilde{z} \\ \tilde{F} & \mapsto & F \\ \tilde{\xi} & \mapsto & x \\ \tilde{t} & \mapsto & a \end{array}\right\} \tag{1.16}$$

Then (1.15) becomes:

$$F_{a,b}(x) = F(x;a,b) = \int_0^x \left[\int_0^y (y-z)^2 g(z)dz\right] dy + a\int_0^x \left[\int_0^y h(z)dz\right] dy + bx \tag{1.17}$$

1.4 Catastrophe Theory Definitions

In this section we state all the relevant definitions and theorems from catastrophe theory from [PS78, pp.157–160] in the one independent variable case (i.e. $n = 1$, the suffix 1 has been removed where it occurred, otherwise all the notation is identical, apart from the definition of J_n^k which has been changed to H^k as it would be ambiguous with the Taylor expansion operator), namely:

Definition 1.4.1 (f is smooth)

$f \in C^\infty[\boldsymbol{R} \to \boldsymbol{R}]$

Definition 1.4.2 (truncated Taylor expansions)

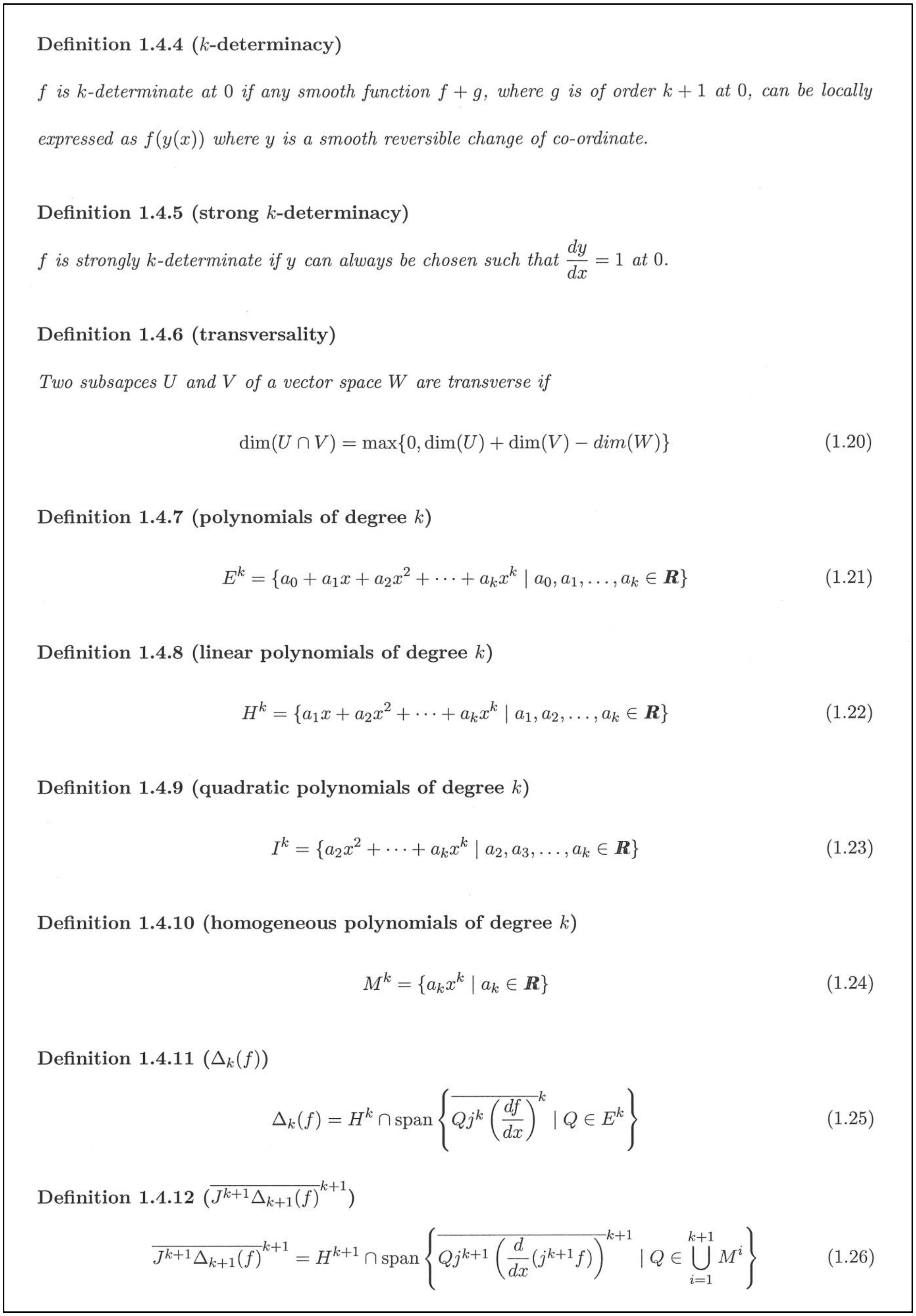
$j^k f$ is the Taylor expansion of f to order k, ie.

$$j^k f = \sum_{r=0}^{k} \frac{x^r}{r!} f^{(r)}(0) \tag{1.18}$$

Alternatively, we may use $\overline{f}^k$ for $j^k f$ when f is a compound expression.

Definition 1.4.3 (linear truncated Taylor expansions)

$$J^k f = j^k f - f(0) \tag{1.19}$$



Definition 1.4.4 (**k-determinacy**)

f is k-determinate at 0 if any smooth function $f+g$, where g is of order $k+1$ at 0, can be locally expressed as $f(y(x))$ where y is a smooth reversible change of co-ordinate.

Definition 1.4.5 (**strong k-determinacy**)

f is strongly k-determinate if y can always be chosen such that $\frac{dy}{dx}=1$ at 0.

Definition 1.4.6 (**transversality**)

Two subsapces U and V of a vector space W are transverse if

$$\dim(U \cap V) = \max\{0, \dim(U) + \dim(V) - dim(W)\} \tag{1.20}$$

Definition 1.4.7 (**polynomials of degree k**)

$$E^k = \{a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \mid a_0, a_1, \ldots, a_k \in \boldsymbol{R}\} \tag{1.21}$$

Definition 1.4.8 (**linear polynomials of degree k**)

$$H^k = \{a_1x + a_2x^2 + \cdots + a_kx^k \mid a_1, a_2, \ldots, a_k \in \boldsymbol{R}\} \tag{1.22}$$

Definition 1.4.9 (**quadratic polynomials of degree k**)

$$I^k = \{a_2x^2 + \cdots + a_kx^k \mid a_2, a_3, \ldots, a_k \in \boldsymbol{R}\} \tag{1.23}$$

Definition 1.4.10 (**homogeneous polynomials of degree k**)

$$M^k = \{a_kx^k \mid a_k \in \boldsymbol{R}\} \tag{1.24}$$

Definition 1.4.11 ($\Delta_k(f)$)

$$\Delta_k(f) = H^k \cap \operatorname{span}\left\{\overline{Qj^k\left(\frac{df}{dx}\right)}^{k} \mid Q \in E^k\right\} \tag{1.25}$$

Definition 1.4.12 ($\overline{J^{k+1}\Delta_{k+1}(f)}^{k+1}$)

$$\overline{J^{k+1}\Delta_{k+1}(f)}^{k+1} = H^{k+1} \cap \operatorname{span}\left\{\overline{Qj^{k+1}\left(\frac{d}{dx}(j^{k+1}f)\right)}^{k+1} \mid Q \in \bigcup_{i=1}^{k+1} M^i\right\} \tag{1.26}$$

Definition 1.4.13 ($\overline{I^{k+1}\Delta_{k+1}(f)}^{k+1}$)

$$\overline{I^{k+1}\Delta_{k+1}(f)}^{k+1} = H^{k+1} \cap \operatorname{span}\left\{ \overline{Qj^{k+1}\left(\frac{d}{dx}(j^k f)\right)}^{k+1} \mid Q \in \bigcup_{i=2}^{k+1} M^i \right\} \tag{1.27}$$

Definition 1.4.14 (codimension)

If U and V are vector spaces and $U \subseteq V$ then the codimension of U in V, written $\operatorname{cod}(U)$ *is given by:*

$$\operatorname{cod}(U) = \dim(V) - \dim(U) \tag{1.28}$$

The codimension of f at 0, $\operatorname{cod}(f)$ *is the codimension of $\Delta_k(f)$ in H^k for any k for which f is k-determinate.*

Definition 1.4.15 (cobasis)

If U and V are vector spaces and $U \subseteq V$ then a cobasis for U in V is a set of vectors $v_1, v_2, \ldots, v_m$ where $m = \operatorname{cod}(U)$, which together with a basis for U yield a basis for V.

Definition 1.4.16 (unfolding)

An r-unfolding of f at 0 is a function

$$\left.\begin{array}{rcl} F : \boldsymbol{R}^{r+1} & \to & \boldsymbol{R} \\ (x, t_1, \ldots, t_r) & \mapsto & F(x; \mathbf{t}) = F_{\mathbf{t}}(x) \end{array}\right\} \tag{1.29}$$

such that $F_{\mathbf{0}}(x) = f(x)$, defined in a region around $(0, \ldots, 0)$.

Definition 1.4.17 (induced unfolding)

A d-unfolding G is induced from F by three mappings, defined in a region about the origin:

$$\left.\begin{array}{rcl} \mathbf{e} : \boldsymbol{R}^{d} & \to & \boldsymbol{R}^{r} \\ (t_1, \ldots, t_d) & \mapsto & (e_1(\mathbf{t}), \ldots, e_r(\mathbf{t})) \end{array}\right\} \tag{1.30}$$

$$\left.\begin{array}{rcl} y : \boldsymbol{R}^{d+1} & \to & \boldsymbol{R} \\ (x, \mathbf{t}) & \mapsto & y(x, \mathbf{t}) = y_{\mathbf{t}}(x) \end{array}\right\} \tag{1.31}$$

$$\gamma : \boldsymbol{R}^{d} \to \boldsymbol{R} \tag{1.32}$$

provided

$$\left.\begin{array}{rcl} G(x;\mathbf{t}) & = & F(y_{\mathbf{t}}(x);\mathbf{e}(\mathbf{t}))+\gamma(\mathbf{t}) \\ ie. \quad G_{\mathbf{t}}(x) & = & F_{\mathbf{e}(\mathbf{t})}(y_{\mathbf{t}}(x))+\gamma(\mathbf{t}) \end{array}\right\} \tag{1.33}$$

Definition 1.4.18 (strong equivalence)

Two r-unfoldings are strongly equivalent to eachother if they can be induced from eachother with $\dfrac{\partial e_i}{\partial t_j}=\delta_{ij}$ *at* 0.

Definition 1.4.19 (versality)

An r-unfolding of f at 0 is versal if all other unfoldings of f at 0 can be induced from it.

Definition 1.4.20 (universality)

An r-unfolding of f at 0 is universal if it is versal and $r=\operatorname{cod}(f)$.

Definition 1.4.21 ($V^k(F)$)

If F is an unfolding of f, set

$$\left.\begin{array}{rcl} v_1^k(F) & = & \dfrac{\partial}{\partial t_1}\left(J^k(F_{t_1,0,\ldots,0})\right) \\ v_2^k(F) & = & \dfrac{\partial}{\partial t_2}\left(J^k(F_{0,t_2,0,\ldots,0})\right) \\ & \ldots & \\ v_r^k(F) & = & \dfrac{\partial}{\partial t_r}\left(J^k(F_{0,\ldots,0,t_r})\right) \end{array}\right\} \tag{1.34}$$

Then

$$V^k(F)=\operatorname{span}\left\{v_1^k(F),\ldots,v_r^k(F)\right\} \tag{1.35}$$

Theorem 1.4.1 (PS78, Theorem 8.1)

f is strongly k-determinate if and only if

$$M^{k+1}\subseteq\overline{I^{k+1}\Delta_{k+1}(f)}^{k+1} \tag{1.36}$$

Proof See [Sie74, TZ76].

Theorem 1.4.2 (PS78, Theorem 8.6)

An r-unfolding F of f, where f is k-determinate, is versal if and only if $V^k(F)$ and $\Delta_k(f)$ are transverse subspaces of H^k.

Proof See [TZ76].

Theorem 1.4.3 (PS78, Theorem 8.6, Corollary)

If f is k-determinate, then a universal unfolding for f may be constructed by choosing a cobasis $v_1, \ldots, v_c$ for $\Delta_k(f)$ in H^k and setting

$$F(x, t_1, \ldots, t_c) = f(x) + t_1 v_1(x) + \cdots + t_c v_c(x) \tag{1.37}$$

Theorem 1.4.4 (PS78, Theorem 8.7)

A versal unfolding F of f is strongly equivalent to the truncated unfolding

$$j^p f(x) + t_1 J^q \frac{\partial}{\partial t_1} F_{t_1,0,\ldots,0} + \cdots + t_r J^q \frac{\partial}{\partial t_r} F_{0,\ldots,0,t_r} \tag{1.38}$$

if f is strongly k-determinate, $k \geq 3$, and

$$\left.\begin{array}{llll} p \geq 2k-3 & q \geq k-2 & \textit{when} & M^{k-1} \subseteq \Delta_{k+1}(f) \\ p \geq 2k-2 & q \geq k-1 & \textit{when} & M^{k} \subseteq \Delta_{k+1}(f) \\ p \geq 2k-1 & \quad\; q \geq k & \textit{when} & M^{k+1} \subseteq \Delta_{k+1}(f) \end{array}\right\} \tag{1.39}$$

At least one of these cases must hold.

Proof See [Mag77].

1.5 Application of Catastrophe Theory

In this section, we shall apply these theorems to the unfolding function $F_{a,b}(x)$ derived in (1.17) in order to transform it into the standard form of the cusp catastrophe A_{+3}:

$$W_{\alpha,\beta}(x) = \frac{x^4}{4} + \alpha\frac{x^2}{2} + \beta x \tag{1.40}$$

Firstly, we must show that F is genuinely an unfolding of a smooth function f.

Lemma 1.5.1 *F is well defined.*

Proof

Let

$$f(x) = F_{0,0}(x) = \int_0^x \left[\int_0^y (y-z)^2 g(z)dz\right] dy \tag{1.41}$$

by using (1.17). We must prove that f is smooth. From (1.16),

$$\begin{aligned} x &= \tilde{\xi} \\ g(\tilde{\xi}) &= -\frac{t_B}{2} c_0^{(3)}(\xi_B + \tilde{\xi}) \end{aligned}$$

Therefore f is smooth provided $c_0^{(3)}$ is smooth. But ψ_0 and c are smooth from (1.2). Therefore c_0 is smooth. Therefore $c_0^{(3)}$ is smooth.

Secondly, according to Definition 1.4.16, we must show that $F_{a,b}(x)$ is defined in a region about $(0, 0)$. This is again guaranteed by the smooth nature of c_0 and the definitions of g and h in (1.16) which go up to make the function $F_{a,b}(x)$.

The lemma is therefore complete. □

Secondly, as we are aiming at inducing the standard unfolding of the cusp catastrophe, we want to apply these theorems with $k = 4$.

Thirdly, we need to show that the smooth function f already derived is strongly 4-determinate by applying Theorem 1.4.1.

Lemma 1.5.2 *f is strongly 4-determinate.*

Proof

From Theorem 1.4.1, f is strongly 4-determinate $\Leftrightarrow \forall a \in \boldsymbol{R}\ \ \exists a_0, a_1, \ldots, a_5 \in \boldsymbol{R}$ such that:

$$ax^5 = \overline{\left[\sum_{r=2}^{5} a_r x^r\right] j^3 \left(\frac{df}{dx}\right)}^{5} \tag{1.42}$$

From (1.16) we have:

$$\frac{df}{dx} = \int_0^x (x-y)^2 g(y)dy \tag{1.43}$$

and by definition we have:

$$j^3\left(\frac{df}{dx}\right) = f'(0) + x\frac{df'}{dx}(0) + \frac{x^2}{2!}\frac{d^2f'}{dx^2}(0) + \frac{x^3}{3!}\frac{d^3f'}{dx^3}(0) \tag{1.44}$$

Therefore

$$j^3\left(\frac{df}{dx}\right) = \frac{x^3}{3}g(0) \tag{1.45}$$

Recalling $g(0) = \frac{t_B}{2}c_0^{(3)}(\xi_B)$. So, as $t_B > 0$, it also follows that $g(0) > 0$.

Thus, assuming $a \neq 0$ (in which case the result is trivial), and by setting $a_2 = \frac{3}{ag(0)}$, and a_3, a_4, a_5 arbitrary we obtain the result. □

Lemma 1.5.3 *F is versal.*

Proof

From Theorem 1.4.2, F is versal when $V^4(F)$ and $\Delta_4(f)$ are transverse subspaces of H^4. Using Definition 1.4.21 with $t_1 = a$ and $t_2 = b$:

$$\begin{aligned}
v_a^4(F) &= J^4\left(\int_0^x \left[\int_0^y h(z)dz\right] dy\right) \\
&= \frac{x^2}{2!}h(0) + \frac{x^3}{3!}h'(0) + \frac{x^4}{4!}h''(0) \\
&= \frac{x^2}{2}c_0^{(1)}(\xi_B) + \frac{x^4}{24}c_0^{(3)}(\xi_B) \qquad (1.46) \\
v_b^4(F) &= J^4 x \\
&= x \qquad (1.47)
\end{aligned}$$

So from Definition 1.4.21:

$$V^4(F) = \left\{\mu\left(\frac{x^2}{2}c_0^{(1)}(\xi_B) + \frac{x^4}{24}c_0^{(3)}(\xi_B)\right) + \nu x \mid \mu, \nu \in \boldsymbol{R}\right\} \tag{1.48}$$

Therefore, as $c_0^{(1)}(\xi_B) < 0$ and $c_0^{(3)}(\xi_B) > 0$:

$$\dim V^4(F) = 2 \tag{1.49}$$

Now, from (1.22), $\dim H^4 = 4$. Also from Definition 1.4.11,

$$\Delta_4(f) = \text{span}\left\{\overline{Qj^4\left(\frac{df}{dx}\right)}^4 \mid Q \in E^4\right\} \cap H^4 \tag{1.50}$$

Now

$$\begin{aligned} j^4\left(\frac{df}{dx}\right) &= \frac{g(0)}{3}x^3 + \frac{g'(0)}{12}x^4 \\ Q &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \end{aligned}$$

for some $a_0, \ldots, a_4 \in \boldsymbol{R}$

Therefore

$$\overline{Qj^4\left(\frac{df}{dx}\right)}^4 = \frac{a_0g(0)}{3}x^3 + \left(\frac{a_1g(0)}{3} + \frac{a_0g'(0)}{12}\right)x^4$$

Therefore, as $g(0) < 0$, whatever the value of $g'(0)$ is we have:

$$\Delta_4(f) = \left\{\alpha x^3 + \beta x^4 \mid \alpha, \beta \in \boldsymbol{R}\right\} \tag{1.51}$$

Therefore

$$\dim \Delta_4(f) = 2 \tag{1.52}$$

Also, as $c_0^{(1)}(\xi_B) < 0$, the x^2 term in (1.48) is always present, so $V^4(F)$ and $\Delta_4(f)$ only intersect at isolated points, ie.:

$$\dim\left(V^4(F) \cap \Delta_4(f)\right) = 0 \tag{1.53}$$

Thus (1.20) is satisfied so $V^4(F)$ and $\Delta_4(f)$ are indeed transverse subspaces of H^4. Therefore F is versal. □

Theorem 1.5.4 *F is strongly equivalent to the unfolding:*

$$T_{a,b}(x) = \frac{t_Bc_0^{(4)}(\xi_B)}{120}x^5 + \frac{t_Bc_0^{(3)}(\xi_B)}{24}x^4 + \frac{ac_0^{(1)}(\xi_B)}{2}x^2 + bx \tag{1.54}$$

Proof

From Theorem 1.4.4, using the first case with $k = 4$ and $q = 5$, provided $M^3 \subseteq \Delta_5(f)$, F is strongly equivalent to the unfolding:

$$T_{a,b}(x) = j^5f(x) + aJ^2\frac{\partial}{\partial a}F_{a,0}(x) + bJ^2\frac{\partial}{\partial b}F_{0,b}(x) \tag{1.55}$$

Firstly, we need to show that $M^3 \subseteq \Delta_5(f)$. From Definition 1.4.11:

$$\Delta_5(f) = \text{span}\left\{\overline{Qj^5\left(\frac{df}{dx}\right)}^5 \mid Q \in E^5\right\} \cap H^5$$

As before,

$$\begin{aligned} j^5\left(\frac{df}{dx}\right) &= \frac{g(0)}{3}x^3 + \frac{g'(0)}{12}x^4 + \frac{g''(0)}{60}x^5 \\ Q &= a_0 + a_1 x + \cdots + a_5 x^5 \end{aligned}$$

for some $a_0, \ldots, a_5 \in \boldsymbol{R}$

Therefore

$$\overline{Qj^5\left(\frac{df}{dx}\right)}^5 = \frac{a_0 g(0)}{3}x^3 + \left(\frac{a_1 g(0)}{3} + \frac{a_0 g'(0)}{12}\right)x^4 + \left(\frac{a_2 g(0)}{3} + \frac{a_1 g'(0)}{12} + \frac{a_2 g''(0)}{60}\right)x^5$$

Therefore

$$\Delta_5(f) = \left\{\alpha x^3 + \beta x^4 + \gamma x^5 \mid \alpha, \beta, \gamma \in \boldsymbol{R}\right\} \tag{1.56}$$

But $M^3 = \{ax^3 \mid a \in \boldsymbol{R}\}$. Therefore $M^3 \subseteq \Delta_5(f)$. Thus we may apply the first case of Theorem 1.4.4 to F. Let us derive the terms on the right-hand-side of (1.55) in turn:

$$\begin{aligned} j^5 f &= \sum_{r=0}^{5} \frac{x^r}{r!} f^{(r)}(0) \\ &= \frac{x^4}{4!} f^{(4)}(0) + \frac{x^5}{5!} f^{(5)}(0) \qquad \text{as } f(0), \ldots, f^{(3)}(0) \text{ are all zero.} \\ &= \frac{g(0)}{12}x^4 + \frac{g'(0)}{60}x^5 \\ &= \frac{t_B c_0^{(3)}(\xi_B)}{24}x^4 + \frac{t_B c_0^{(4)}(\xi_B)}{120}x^5 \end{aligned} \tag{1.57}$$

$$\begin{aligned} J^2 \frac{\partial}{\partial a} F_{a,0} &= J^2 \int_0^x \left[\int_0^y h(z)dz\right] dy \\ &= \frac{x^2}{2} h(0) \\ &= \frac{c_0^{(1)}(\xi_B)}{2} x^2 \end{aligned} \tag{1.58}$$

$$\begin{aligned} J^2 \frac{\partial}{\partial b} F_{0,b} &= J^2 x \\ &= x \end{aligned} \tag{1.59}$$

Hence by adding $j^5 f$, $aJ^2 \frac{\partial}{\partial a} F_{a,0}$ and $bJ^2 \frac{\partial}{\partial b} F_{0,b}$ we obtain the result. □

Theorem 1.5.5 *$W_{\alpha,\beta}(x)$ may be induced from $T_{a,b}(x)$.*

Proof

The first step is to show that $t(x) = T_{0,0}(x)$ is 4-determinate. We have already shown that f is strongly 4-determinate. As t is merely a truncation of f it follows that t is also strongly 4-determinate. As t is strongly 4-determinate, it is also 4-determinate.

Therefore, by Definition 1.4.4 with $k = 4$, there exists a smooth reversible function $y(x)$ such that t can locally be expressed as

$$t(y(x)) = \frac{t_B c_0^{(3)}(\xi_B)}{24} x^4 \tag{1.60}$$

It is straightforward to show that:

$$x = y(x)\left(1 + \frac{c_0^{(4)}(\xi_B)}{5c_0^{(3)}(\xi_B)} y(x)\right)^{\frac{1}{4}} \tag{1.61}$$

(where the real fourth root near to 1 is taken near to $y = 0$.)

Therefore we may construct a 2-unfolding $U_{a,b}(x)$ which may be induced from $T_{a,b}(x)$ in the straightforward way to give:

$$\begin{aligned} U_{a,b}(x) &= T_{a,b}(y(x)) \\ &= \frac{t_B c_0^{(3)}(\xi_B)}{24} x^4 + a\frac{c_0^{(1)}(\xi_B)}{2} y(x)^2 + by(x) \end{aligned} \tag{1.62}$$

Next, let us set

$$\frac{t_B c_0^{(3)}(\xi_B)}{24} z^4 = \frac{1}{4} x^4 \tag{1.63}$$

Again, taking the real positive fourth root, we obtain:

$$z(x) = \left(\frac{6}{t_B c_0^{(3)}(\xi_B)}\right)^{\frac{1}{4}} x \tag{1.64}$$

Thus we introduce another induced unfolding $V_{a,b}(x)$, where:

$$\begin{aligned} V_{a,b}(x) &= U_{a,b}(z(x)) \\ &= \frac{t_B c_0^{(3)}(\xi_B)}{24} z(x)^4 + \frac{a c_0^{(1)}(\xi_B)}{2} y(z(x))^2 + b(y(z(x)) \\ &= \frac{x^4}{4} + \frac{a c_0^{(1)}(\xi_B)}{2} y\left(\left(\frac{6}{t_B c_0^{(3)}(\xi_B)}\right)^{\frac{1}{4}} x\right)^2 + by\left(\left(\frac{6}{t_B c_0^{(3)}(\xi_B)}\right)^{\frac{1}{4}} x\right) \end{aligned} \tag{1.65}$$

(where y operates on the expressions in the following brackets.)

We may now apply Theorem 1.4.3 to finally reduce $V_{a,b}(x)$ to the required form. Firstly, we set

$$v(x) = V_{0,0}(x) = \frac{1}{4}x^4 \tag{1.66}$$

Clearly, v is 4-determinate. Therefore, from Theorem 1.4.3, we may construct a universal unfolding for v by choosing a suitable co-basis for $\Delta_4(v)$ in H^4.

Now, from Definition 1.4.11,

$$\Delta_4(v) = \text{span}\left\{\overline{Qj^4\left(\frac{dv}{dx}\right)}^4 \mid Q \in E^4\right\} \cap H^4 \tag{1.67}$$

But

$$\begin{aligned} \frac{dv}{dx} &= x^3 \\ Q &= \sum_{r=0}^{4} a_r x^r \quad \text{for some} \quad a_0, a_1, \ldots, a_4 \in \boldsymbol{R} \end{aligned}$$

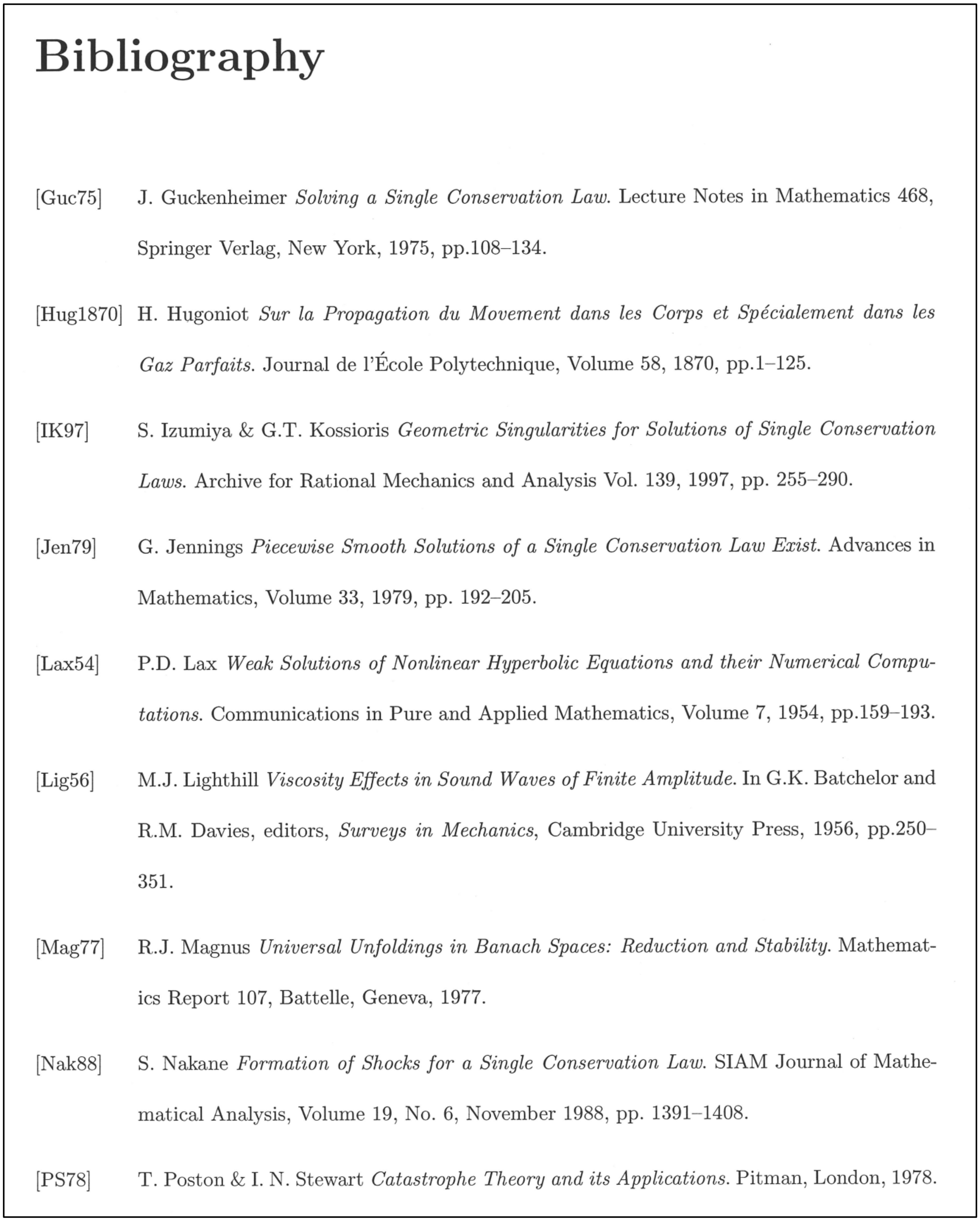
Therefore

$$\overline{Qj^4\left(\frac{dv}{dx}\right)}^4 = a_0x^3 + a_1x^4$$

So from Definition 1.4.15, a cobasis for $\Delta_4(v)$ is $\left\{x, \dfrac{x^2}{2}\right\}$. Therefore, applying Theorem 1.4.3 we obtain the following universal unfolding for $V_{a,b}$:

$$W_{\alpha,\beta}(x) = \frac{x^4}{4} + \alpha\frac{x^2}{2} + \beta x$$

This completes the proof. □



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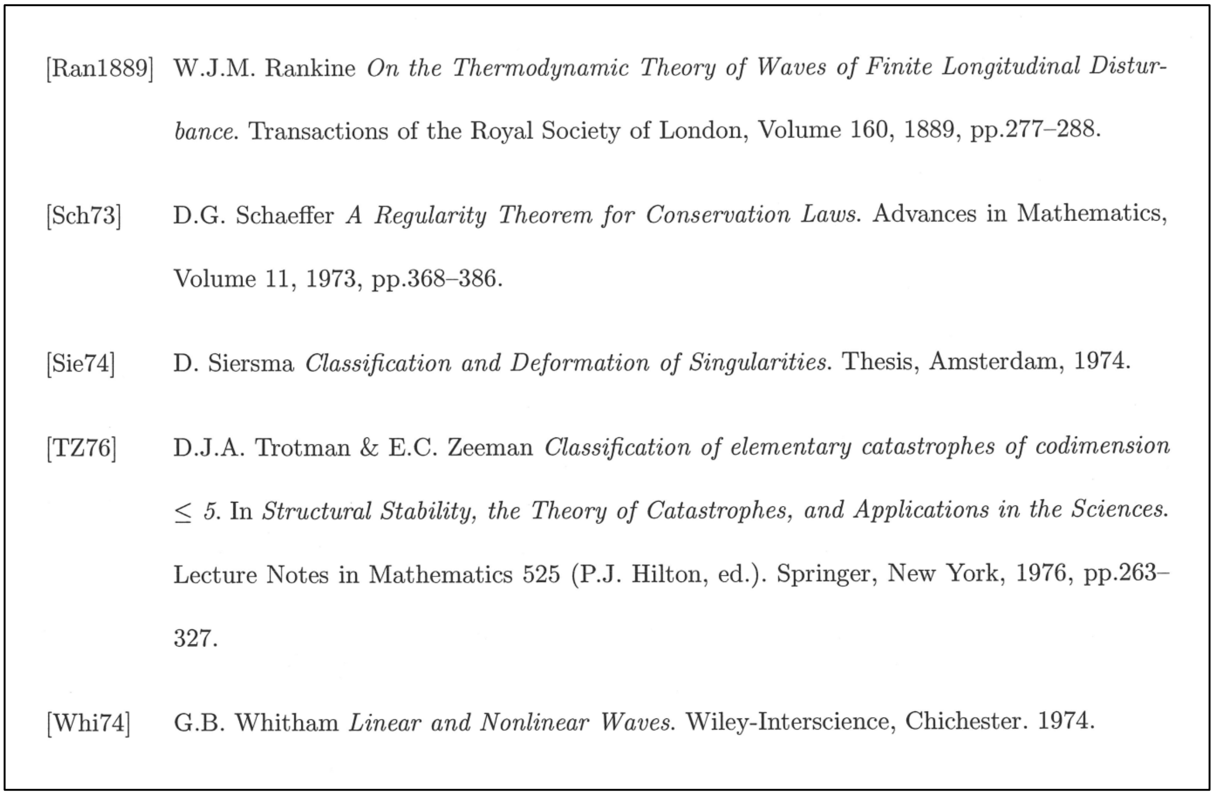

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