Research Article

Eliciting Students' Critical Thinking to Connect Phenomena and Representations on Context-Based Writing-to-Learn Assignments in Mathematics

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Writing-to-learn (WTL) pedagogies support students' conceptual learning and disciplinary thinking (Anderson et al., 2015; Gere et al., 2019). A variety of articles across science, technology, engineering, and mathematics (STEM) education research disciplines demonstrate how WTL can be used to elicit evidence of how students both engage with course content and demonstrate disciplinary thinking skills (Balgopal & Wallace, 2013; Cox et al., 2018; Finkenstatedt-Quinn et al., in press). In the context of undergraduate mathematics education, a key aspect of disciplinary thinking that students must develop is the ability to connect real-world problems to mathematical representations (such as equations or graphs) in order to solve problems and translate the results into real-world solutions. This process of connecting real-world problems to mathematic representations is called mathematical modeling (Czocher et al., 2020; Stillman et al., 2020), which corresponds to the practice of "developing and using models" identified in the Next Generation Science Standards for improving STEM education (NGSS Lead States, 2013). Not only is modeling an important process for students to learn, but it can support the development of students' critical thinking skills (Asempapa, 2015; Lesh & Zawojewski, 2007; Nicholes & Lukowski, 2021). In the undergraduate curriculum, one course with mathematical modeling at its center is differential equations, which focuses on the use of equations to model complex systems that change with time; these equations can be solved using calculus to afford solutions with real-world meaning (Brannan & Boyce, 2015).

Existing research in mathematics education examines the process whereby novices and experts engage in mathematical modeling and offers suggestions for instructors to support students in developing this skill (Ärlebäck et al., 2013; Crouch & Haines, 2007; Schleppegrell, 2007; Stillman & Brown, 2021). Closely related to mathematical modeling is representational competence, which is the ability to use representations (e.g., equations or graphs) to describe and explain phenomena (Kozma & Russell, 2005). Representational competence represents a more narrowly defined skillset necessary for mathematical

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modeling; it specifically pertains to aspects of the modeling process that involve translating between real-world scenarios and the relevant representations to derive real-world solutions. Many studies examine how students develop and demonstrate representational competence across STEM contexts (Kozma & Russell, 2005; Moore & Thompson, 2015; Rasmussen, 2001). Both mathematical modeling and engaging with representations require critical thinking because they involve analyzing representations of real-world problems to make judgements and develop arguments to derive real-world solutions and draw conclusions (Abassian et al., 2020; Kozma & Russell, 2005). The present study translates the existing research on mathematical modeling and representational competence into classroom practice through the design and implementation of a WTL assignment. We examine students' responses to the assignment, with the goal of identifying how they use and analyze representations as they engage in critical thinking through a mathematical modeling process.

Mathematical Modeling and Critical Thinking

Models are important throughout STEM disciplines for visualizing, explaining, and making predictions about phenomena (Gilbert, 2005). The ability to develop and use models to explain and predict real-world phenomena requires critical thinking and is a key educational outcome for STEM education broadly (NGSS Lead States, 2013) and increasingly for mathematics education in particular (Stillman et al., 2020).

Incorporating the modeling process into mathematics classrooms is also thought to support conceptual learning (Czocher, 2017; Zbiek & Conner, 2006) as well as student affect and self-efficacy (Czocher et al., 2020; Schukajlow et al., 2021; Zbiek & Conner, 2006). The mathematical modeling process typically includes understanding the real-world problem, simplifying the situation, transforming the problem into a mathematical model (such as an equation or graph, through a process called mathematization), working mathematically to produce mathematical results, and translating mathematical results into the real situation for interpretation and validation (Abassian et al., 2020). The modeling process is often conceptualized as recursive, in that going through the cycle several times may be needed. The process requires understanding the information contained within the mathematical representations (i.e., the external models or visualizations of mathematic concepts, such as formulas and graphs) and considering how the representations align with the scenario being modeled. However, research indicates that students' abilities to work with and reason about representations are still developing as they progress through post-secondary education (Ärlebäck et al., 2013; Herbert & Pierce, 2011). Hence, there is a need to support students' understanding of representations in the context of mathematical modeling.

Students' consideration of the real-world problem, or phenomenon, being modeled is also a critical aspect of the modeling process, yet it is another component of modeling where students may benefit from additional training. In a study of mathematics majors in their last year of undergraduate coursework, Crouch and Haines (2007) observed that while students were relatively adept at modeling, some novice-like behaviors were still present; these often related to a lack of consideration of the real-world phenomenon during the modeling process. Similarly, Stillman and Brown (2021) found that, when working with data sets, students focused on modeling the specific data set at hand rather than modeling the given phenomenon or scenario of which it was representative. Relatedly, students can have difficulty using language to engage in mathematical discourse generally (Schleppegrell,

2007) and, with respect to modeling, specifically when they have to colloquially describe what is being depicted by a representation in relation to a phenomenon (Ärlebäck et al., 2013). Considered as a whole, findings from prior research indicate that it is important to provide students with opportunities to engage with representations in modeling real-world phenomena and to characterize how students think about representations, both individually and in connection to one another, as they engage in the modeling process. Because working with representations is central to the modeling process, the focus of this article is on exploring how students think critically about mathematical representations in the context of an undergraduate-level introductory differential equations course.

Representational Competence and Critical Thinking

Not only are representations important for modeling, but the ability to create and think critically about representations is key to STEM disciplines and student learning in its own right (Gilbert, 2005; National Research Council, 2012). Within mathematics specifically, representations and visualizations can play a fundamental role in critical thinking, problem solving, reasoning, and mathematical thinking (Arcavi, 2003; Janvier, 1987). Like modeling, representations have been identified as a standard that should be included in mathematics education (National Council of Teachers of Mathematics, 2000). The existing research focuses primarily on students' sense-making, with minimal focus on how they interpret and use representations as they engage in modeling behaviors. Findings indicate that representations can support students' understandings of complex mathematical principles but that students do not always fully grasp the meaning captured by a representation and often interpret representations differently from experts (David, 2018; David et al., 2019; KarimiFardinpour & Gooya, 2018; Moore, 2016; Moore & Thompson, 2015; Parr, 2021; Rasmussen, 2001). As students move to post-secondary mathematics courses, they are still developing mathematical thinking, such as covariational reasoning; recognizing functions as solutions; and identifying the relationship between functions and graphs (Carlson et al., 2002; David, 2018; David et al., 2019; Parr, 2021; Rasmussen, 2001). The research findings about students' abilities to interpret representations relates to a larger need in mathematics education for research on pedagogies that can provide students with opportunities to engage in critical thinking about representations (Parr, 2021; Presmeg, 2006; Rasmussen, 2001; Rowland & Jovanoski, 2004).

In this work, we operationalize critical thinking about representations by considering the role of *representational competence*. Kozma and Russell (2005) articulated the skills necessary for representational competence, which include the ability to

- use representations to describe phenomena,
- explain why a representation is appropriate for a specific use,
- analyze the features of a representation (such as the curves on a graph),
- describe how different representations of the same phenomenon can be used for different purposes,
- identify connections and relationships across different representations,
- understand that representations are distinct from the represented phenomena, and

- use features of representations in social situations as evidence to support claims.

Because of its emphasis on communication, the final skill is particularly relevant to WTL assignments, to which students' responses can provide evidence of their representational competence.

Supporting Critical Thinking with Representations Using Writing-to-Learn

The importance of representations for modeling real-world phenomena and communicating mathematical ideas supports the use of WTL pedagogy (Anderson et al., 2015; Bangert-Drowns et al., 2004; Gere et al., 2019; National Council of Teachers of Mathematics, 2000), which specifically promotes the use of writing assignments for students to reach conceptual learning goals. Broadly, WTL prompts students to articulate their thinking and make it visible through their writing. Many studies have indicated the value of WTL assignments for eliciting evidence of students' conceptual understanding in STEM courses, including statistics, materials science, chemistry, and biology (see, e.g., Brandfonbrener et al., 2021; Finkenstaedt-Quinn et al., 2017; Finkenstaedt-Quinn et al., 2019; Finkenstaedt-Quinn et al., 2020; Finkenstaedt-Quinn, Polakowski, et al., 2021; Gupte et al., 2021; Halim et al., 2018; Moon et al., 2018; Moon et al., 2019; Petterson et al., 2022; Schmidt-McCormack et al., 2019; Shultz & Gere, 2015; Watts et al., 2020; Watts et al., 2022), in addition to mathematics courses, including foundational mathematics and calculus (Elder & Champine, 2016; Van Dyke et al., 2015). The studies across disciplines, and in mathematics specifically, evaluated students' writing in relation to course outcomes to explore students' understanding and critical thinking. A recent meta-analysis additionally indicates the value of implementing writing to support students' conceptual learning in mathematics (Bicer et al., 2018). Hence, the prior research provides a basis for (1) implementing WTL to support students' representational competence in a differential equations course and (2) analyzing students' writing to identify aspects of students' representational competence.

Using WTL to support students' engagement in modeling and to elicit evidence of their representational competence centers on one of the key components of effective WTL assignments: the inclusion of a meaning-making task. Meaning-making tasks require students to move beyond reporting known information and towards constructing and transforming knowledge (Gere et al., 2019). They often present a rhetorical situation that gives students a role to assume and audience to write for in an authentic, real-world context for applying conceptual knowledge. Examples of meaning-making tasks can include analyzing or evaluating data or constructing arguments or explanations. By situating the writing task within an authentic context, WTL assignments provide a means for engaging in critical thinking by requiring students to connect real-world phenomena to mathematical concepts. In other words, WTL assignments afford students the opportunity to engage with the modeling process, and the artefacts of students' written responses can serve as evidence of their representational competence.

The goal of this study is to identify students' representational competence skills as evidenced through their written responses to a set of three WTL assignments in a differential equations course. It is necessary to understand students' representational competence because it is central to the broader skill of engaging in mathematical modeling. Extending the literature by focusing specifically on understanding students' representational competence can serve to better inform instructional decisions focused on supporting students at each stage of the modeling process. As such, this study is guided by the following research question:

> How do students in a differential equations course use and analyze representations as they engage in critical thinking about mathematical models in their responses to WTL assignments?

Theoretical Framework

This research is grounded in the sociocultural theory of writing, which conceptualizes writing as both an individual and social activity (Prior, 2006). This theory considers all forms of writing to be collaborative in that they are informed by social constructs and contexts. Social constructs that guide the act of writing range from the language being used, and associated grammatical rules, to the modality in which a text is produced. Perhaps the most pertinent social construct that influences writing is genre, which reflects disciplinary conventions and ways of thinking (Bazerman, 2009). Social context includes other people who may actively co-author a text or external representations of knowledge that inform the content of the text. These components of the sociocultural theory of writing provide explanatory power for interpreting and understanding student writing by recognizing the social constructs and contexts that influence it.

For this study, the sociocultural theory of writing allows us to examine the artefacts of students' writing to interpret how students demonstrated their representational competence as they engaged in the modeling process. The WTL assignments examined herein incorporated collaborative writing in two ways: students drafted their initial and revised responses in pairs; secondly, their writing underwent peer review, providing them with feedback as an external source of knowledge. This structure reflects the scientific practice of collaboration and peer review, wherein scientists refine and adjust their ideas and how they represent knowledge, and enables us to view students' final responses to the WTL assignments as the best representation of their critical thinking.

Methods

The goal of this study is to investigate the utility of WTL for eliciting students' representational competence as they engage in critical thinking about mathematical models. To achieve this goal, we examined students' responses to a set of three WTL assignments within a differential equations classroom. Through qualitatively analyzing students' responses to the WTL assignments, we seek to provide exploratory insight by characterizing the varied ways in which students use and analyze representations as they engage in mathematical modeling through writing. In the following sections, we describe our positionality, the setting and participants, the WTL assignments and implementation, and our data analysis process.

Positionality Statement

This study is part of a greater research effort under a WTL program at the University of Michigan, called MWrite, which is focused on studying the effectiveness of WTL for supporting conceptual learning and disciplinary thinking. As the MWrite program works directly with faculty to design and implement WTL assignments, it is important for us to acknowledge our positionality with respect to this study. None of the researchers in this study is affiliated with the differential equations course from which we gathered data. However, we all have connections to the MWrite program. Specifically, we are the program manager (SFQ), a recent doctoral graduate (FMW), and two undergraduate students who previously supported the implementation of MWrite in a statistics course (CR and NB). These roles gave us a familiarity with the potential benefits of WTL and existing research in ways that may have guided this study (e.g., the focus on students' writing as a representation of their knowledge).

Setting and Participants

The study was conducted at the University of Michigan in an introductory differential equations course. Enrollment in the course is typically 500–550 students each semester. The class is divided into three lectures and one lab per week; the labs entail smaller sections of students meeting to apply concepts discussed in the lecture. The course is taken primarily by sophomores and juniors; it is intended for and primarily taken by engineering majors rather than mathematics majors. The material for the course is presented with a focus on how students interpret differential equations and their solutions in real-world contexts. Students are evaluated through two exams and a final, five labs, weekly online homework, and five written homework assignments. The writing assignments are associated with the labs in the course; three of the writing assignments are MWrite WTL assignments (Finkenstaedt-Quinn, Petterson, et al., 2021), while the other writing assignments are shorter reflections on the associated lab material. The three MWrite WTL assignments are the focus of this study.

WTL Assignment Descriptions and Implementation

For the three labs that were followed by WTL assignments, students were presented with detailed instructions that included relevant MATLAB commands, background information, and a series of exercises that walked them through the modeling process. The assignment descriptions provided students with a rhetorical situation, audience, and genre to guide their writing, situating the mathematics in a real-world context and providing them with a meaning-making task through which to engage in the modeling process (Finkenstaedt-Quinn, Petterson, et al., 2021). The first WTL assignment required students to model the growth of cancerous tumors under various conditions using the Gompertz equation, a first-order differential equation, and to consider when using approximations to the equations is appropriate. The second WTL assignment required students to model the behavior of lasers under various conditions, using a given system of equations. The third WTL assignment introduced students to the complexities of climate modeling through the use of the Lorenz equations, a three-dimensional system of equations. Hereafter, the WTL assignments will be referred to as the *Cancer*, *Lasers*, and *Climate* assignments, respectively. The WTL assignments are presented in Appendix A.

Students worked with a partner on the labs and associated WTL assignments. Implementation of the WTL assignments took place in three stages. Students submitted drafts with their partners approximately a week after the lab had been completed. Students then engaged in peer review, facilitated by an automated tool for distributing peer review assignments. During peer review, students provided responses to four content-focused, prompt-specific criteria to structure their feedback for each assignment they reviewed. Peer review assignments were completed individually, in that each student read and provided feedback on typically three drafts from different groups. Following peer review, the students worked again with their partner to revise their drafts. Assessment of the WTL assignments was independent of the presented analysis.

Data Collection and Analysis

Students' final drafts were collected for the three WTL assignments from the winter 2019 term of the course. (The Institutional Review Board approved the study as exempt from review [HUM00115139].) The instructor of the course randomly selected and de-identified approximately 20 revised responses from each assignment (approximately 10% of the responses for each assignment) for the research team after course grades had been assigned. In alignment with the sociocultural theory of writing, we chose to focus on students' final drafts for the analysis in order to identify evidence of their representational competence after engaging in the entire collaborative writing process with peer review. Because the responses are from pairs of students and may not reflect any individual student's representational competence, the final, revised drafts reflect students' critical thinking and representational competence as a function of the social nature of writing in professional contexts. The writing was analyzed through qualitative coding and thematic analysis, as described in the following section.

Qualitative Analysis

A coding scheme was developed and applied to student writing, followed by thematic analysis of the dataset to identify prevalent themes (Braun & Clarke, 2006; Watts & Finkenstaedt-Quinn, 2021). The scheme was developed through simultaneous deductive and inductive coding with constant comparative analysis (Miles et al., 2014). We used the representational competence framework (Kozma & Russell, 2005) as an initial set of deductive codes, and we also inductively coded for features present in students' responses that were not within the conceptual framework but were pertinent for more fully characterizing students' representational competence in this context (e.g., when students discussed how they generated the graphs in the lab). This development process began with two researchers (CR and NB) independently reading a subset of students' responses while applying the deductive codes and creating inductive codes. Throughout the process, the full research team met to refine the full set of codes into a single coding scheme. The process continued through multiple rounds of coding and team discussions until reaching saturation and deciding upon a finalized coding scheme.

The final coding scheme contained four sections: introducing representations, discussing the appropriateness/limitations of representations, usage of representations, and general assignment features. The introducing representations section contained codes connected to the introduction of graphs, formulas, and variables. The appropriateness/ limitations section contained codes related to the appropriateness and limitations of the representations. The usage of representations section contained codes related to how graphs and formulas were used in connection with the given phenomenon. Lastly, the general assignment features section included codes that captured references to the assignment itself or its overall purpose or references to using MATLAB to create graphs and figures. The full coding scheme is presented in Appendix B. Codes were applied to students' work at the sentence level, and multiple codes could be applied to each sentence (Krippendorff, 2004).

Two researchers (CR and NB) independently analyzed additional student responses with the finalized coding scheme and met to discuss their application of codes. Because of the length and complexity of the responses, each response was discussed between researchers to arrive at a consensus for the final set of codes applied, in alignment with Campbell et al.'s (2013) recommendations for analyzing complex data sources (Watts & Finkenstaedt-Quinn, 2021). Furthermore, the responses used in the development of the coding scheme were re-analyzed with the finalized version of the coding scheme to reach a consensus on the applied codes. In total, 30 responses were analyzed (10 from each WTL assignment).

After all 30 documents were coded, the results were used for thematic analysis to identify common themes across students' responses in correspondence with the guiding research question (Braun & Clarke, 2006). The codes were sorted into potential themes based on their alignment with each other and the research question. The potential themes were reviewed and refined through returning to the coded data and re-reading full responses to ensure the themes aligned with the data. Through ongoing, reiterative analysis and discussions with the research team, the themes were fully refined through the process of producing the results and discussion sections of the manuscript.

Results

The goal of this study is to identify students' representational competence skills, with respect to mathematical modeling, as elicited by the set of three WTL assignments. Students' collaborative responses serve as an external representation of their representational competence and as an indication of their abilities to use representations to communicate mathematically after engaging in the sociocultural act of writing. Overall, the assignments elicited the skills relevant to representational competence. Our thematic analysis process resulted in the identification of three themes: (1) students connected representations to the phenomenon they were describing and analyzed the representations mathematically, though to different extents for each representation; (2) students included multiple mathematical representations in their writing, but varied in whether and how they made connections between representations; and (3) students included both the limitations and appropriateness of representations for modeling the phenomena of interest. These themes align with the representational competence skills outlined by Kozma and Russell (2005), and they are presented in the following subsections. Exemplars for illustrating trends in the analysis are provided throughout, using pseudonyms for the student pairs.

Across All Three Assignments, Students (1) Connected the Representations to the Phenomenon They Were Describing and (2) Analyzed the Representations Mathematically, Though to Different Extents for Each Representation

When introducing a representation, students either connected the representation to the phenomenon they were describing or analyzed the mathematical features of the representation salient to the problem they were solving. During our analysis process, it became apparent that the extent to which students made connections to the phenomenon and/or analyzed the features of the representations differed depending on where the description occurred in their responses (i.e., the introduction, body, or conclusion). In general, students incorporated more connections to the phenomena in the introduction and conclusion relative to the body paragraphs, which tended to contain most of the analysis.

In the introduction, students tended to present the relevant formulas and explain what each variable represented in the context of the phenomenon of interest. For example, in response to the *Lasers* assignment, one student pair presented the formulas and wrote,

> In the provided equations above N is the population inverse of the number of atoms in the laser and P is the laser's intensity. These equations model the number of atoms in each of the first three energy states. (Morrison & Toni, *Lasers*)

In an example from the *Climate* assignment, one student pair described how the Lorenz system can be used to model climate patterns:

> The Lorenz system is a three dimensional matrix that describes the intensity of motion of particles in the fluid (x), the difference in temperature of particles moving in the spatial z-axis (y), and the distortion of particles along the spatial z-axis (z). (Whitney & Emerson, *Climate*)

That students were able to connect the variables within formulas to their meaning in the context of the phenomenon of interest indicates that the students were generally successful at interpreting the features of the mathematical formulas, a necessary skill for representational competence (Kozma & Russell, 2005).

As students moved into the body paragraphs of their responses, they began to analyze graphs and typically made fewer connections to the phenomenon they were describing. Students spent the majority of their writing analyzing the graphs individually after first referencing them. For example, one student pair analyzed specific features of a single graph depicting laser efficiency:

> Notice that as it continues to grow from 1.5 to 3 we see an appropriate increase in the intensity, while N seems to stay roughly the same. (Austen & Jane, *Lasers*)

This description demonstrates that as with their interpretation of the features of formulas, students were successful in identifying and analyzing features of the graphs. Students also identified patterns across multiple graphs. For example, the same student pair wrote,

> These graphs both approach the value (A-1), which is, of course, the P value of the critical point (1, A-1) of system (1) and the value we linearized the system about. (Austen & Jane, *Lasers*)

Notably, as evident in these sample responses, students would not often provide much explanation of what the graphs were modeling in terms of the phenomenon within the body paragraphs of their responses. In the above example, when analyzing the graphs, the students did not explain what this mathematical statement meant with regard to the behavior of the laser. In another example, one student pair wrote:

> With the helpful addition of the light blue zero line we can clearly see the presence and change of roots with varying r. In the cases where r is less than 1.3456178 the roots are all real. (Whitney & Emerson, *Climate*)

These students explained how roots change with varying *r* but did not connect the result to what it indicates in terms of weather systems. In general, it was common across students' writing to focus on analyzing features of the graphs mathematically without connecting these features to the phenomenon of interest.

In contrast, there were some students who successfully connected the representations to the phenomenon of interest within their analysis sections. For example, one student pair wrote:

> Notice that for A < 1, no oscillatory behavior presents itself and P quickly approaches zero, this means that the laser would not be outputting very many photons at all. (Austen & Jane, *Lasers*)

This student pair connected the mathematical idea of *P* approaching zero to what that means in terms of laser output. The finding that most students connected representations to the phenomena of interest within the introduction, but fewer students did so when analyzing the graphical representations, suggests that only some student pairs may have identified the importance of connecting specific representational features to the phenomena of interest during the analysis stage. However, the fact that some students did successfully connect their analysis to the phenomenon of interest indicates that students are capable of achieving this learning goal, though some students may require more scaffolding to guide their thinking on when to make the connections.

Students were most successful at connecting their analysis of graphs to the phenomenon of interest within their conclusions. In general, students' conclusions synthesized their analyses and nearly always included statements that related the phenomena of interest to the representations. For example, one student pair used the graph to make a clear conclusion regarding laser efficiency:

> This corresponds to the critical point (1, A-1), and provides a mathematical rationale for why the intensity continues to grow with a while the population inversion stays fairly constant. (Austen & Jane, *Lasers*)

Relatedly, students used graphs to support their concluding statements more often than they used formulas. For example, the same student pair stated,

> By modeling with the linearized equations, we find that we can predict the photon emission very well. We also see that the equilibrium solution gives us an insight on how the intensity of the laser will behave for various parameters. (Austen & Jane, *Lasers*)

As an example of students making connections between graphs and the phenomenon in the conclusion of their writing for the *Climate* assignment, one student pair wrote:

> The Lorenz System is a very useful tool when it comes to predicting temperature change. Despite its complexities, crucial information about changes in temperature can be found by looking at the [graph's] behavior around the system's critical points. This behavior can be altered with different values of 'r' (a constant proportional to the differences in temperature between layers). Looking at some of the most notable 'r' values can give us key insight into weather patterns granting us the ability to forecast the weather further into the future. (Whitney & Emerson, *Climate*)

In this example, the student pair connected the features of different graphs (the behavior at the critical points, which varies with different *r* values) to the phenomenon of interest to make their conclusion. Students' choices to focus on the graphs rather than formulas in the conclusion may indicate that they find the graphs more suitable for directly relating results back to the phenomena of interest. However, the statements made in the conclusion were often more general compared to the specificity of the statements in the introduction, in that students did not make connections to specific features of the graphs.

Students Included Multiple Mathematical Representations in Their Writing, but They Varied in Whether and How They Made Connections Between the Representations

Across all three assignments, students rarely compared formulas to one another and instead presented them individually. Students also rarely compared the graphs to one another (but more so than with the formulas), though they often included multiple graphs and demonstrated a clear understanding of them individually through detailed analysis, as described in the previous section. Hence, it appears that some student pairs may have lacked an understanding of either how to make connections within representations of the same type (e.g., formula to formula or graph to graph) or the usefulness thereof when modeling a phenomenon.

When students did compare graphs, the analysis was typically brief and less detailed than their analysis of graphs separately. For instance, one student pair wrote,

> The graphs start out fairly differently, but there are some notable similarities. As shown by the figure above, both oscillations seem to have crests and troughs in roughly the same locations despite the amplitudes being very dissimilar. Most importantly to note however, is that, as time goes on, the behavior of the graphs begin to mimic each other very closely in the sense that they both appear to approach the value 2. (Austen & Jane, *Lasers*)

While the students compared the two graphs, they did not draw conclusions from their comparisons. For example, the students did state the similarities and differences between the graphs (i.e., the similarity in terms of crests and troughs of the oscillations, but with different amplitudes) but left the interpretation of this comparison for the reader to infer. These findings suggest that students are capable of integrating graphs into their writing by analyzing and making conclusions based on individual graphs, but that some students may not take the next step of making connections and comparisons between graphs within their analysis. Rather, the connections student pairs made between graphs tended to be more broadly stated within their conclusions.

Students did, however, make connections and comparisons between formulas and graphical representations. Some student pairs were very clear in demonstrating that the graphs were based upon the formulas, such as in the following example:

> The following information is deduced from the equations above and Figure 2 below. (Davis & Lydia, *Cancer*)

Comparatively, other student pairs only alluded to the relationship:

> From Figure 2, the tumor growth is predicted to have a logistic growth that eventually approaches y = K. (Carson & Anne, *Cancer*)

While it was clear from the student pair's introduction that the values *y* and *K* are part of the Gompertz equation that models tumor growth, the student pair did not explicitly connect the values previously described with respect to the formula. Additionally, in this case students could have derived the logistical growth relationship directly from the graphical representations without any consideration of the formula.

Students' Writing Included Both the Limitations and Appropriateness of Representations for Modeling the Phenomena of Interest

Another key skill for representational competence is recognizing the limitations of representations and the appropriateness of when to use representations to model a phenomenon (Kozma & Russell, 2005). Students discussed the limitations of the representational models across all three assignments, though to differing extents. In the *Lasers* assignment, students were only indirectly pointed to limitations of the models, whereas in the *Climate* assignment, the unpredictability of the phenomenon is made much more explicit and points students toward the limitations. Thus, the differences in students' discussions are likely explained by the assignments themselves, both in the phenomena being modeled and how the assignment descriptions describe the ease of modeling the phenomena. When discussing limitations, students most typically did so in the analysis section of their responses. They often directly related the limitations to the formulas and graphs rather than to the phenomena of interest. For example, one student pair presented the Taylor expansion formula modeling tumor growth and followed it with the sentence,

> However, these simplifications are only good approximations to the original model under certain circumstances. (Davis & Lydia, *Cancer*)

In this example, the student pair discussed the limitation of the specific formula without explicitly relating it to the phenomenon at hand. In contrast, some student pairs did discuss the limitations of a representation with respect to the phenomenon of interest:

> If the model is simplified by assuming a small tumor size, it will not be as accurate as we expand the Gompertz equation, as a larger tumor would. (Nelson & Maggie, *Cancer*)

While this student pair connected the limitation of a specific representation to the phenomena of interest, others did not—this finding indicates that this representational competence skill is one that can be achieved by students, though they may require more instructional support.

As with their discussion of limitations, students incorporated discussions of the appropriateness of representations for modeling the phenomena across all three assignments; however, they tended to write about the appropriateness in their conclusions, often in broad terms. For example, one student pair wrote:

> Through using a smaller tumor size we found that the long-term behavior is the same throughout the expansions, however as we imputed a larger tumor size, we found that the accuracy of the expansions are much greater than with the smaller tumor size. (Nelson & Maggie, *Cancer*)

As shown here, when students made statements regarding the appropriateness of the representations, it was often part of an overview that involved comparing or summarizing multiple graphical representations. In another example, after summarizing multiple graphs modeling tumor growth, one student pair wrote,

> Thus, it can be said that Taylor approximations are most accurate in determining the duration of tumor growth to its maximum size, as well as what that size is. (Davis & Lydia, *Cancer*)

In this example, the pair explained the accuracy of the Taylor approximations for modeling tumor growth from looking at multiple graphical representations and related the accuracy of it directly back to the phenomenon. The fact that student pairs described the appropriateness of the representations for modeling the phenomenon primarily when comparing graphical representations may indicate that it is through making comparisons that they can more easily identify or articulate the appropriateness of a model for representing a phenomenon. This finding also aligns with students' propensity to make broader connections between graphical representations in their conclusions, as described previously.

Discussion

Overall, the analysis indicates that students demonstrated a majority of the skills of representational competence in their writing. The results suggest that, when working collaboratively, students can successfully connect representations to the phenomenon of interest in their writing and that they do so primarily in the introduction and conclusion of their responses. This finding may indicate that the assignments support critical thinking about mathematical models. For example, the results suggest that students are engaged in simplifying and transforming from the phenomenon to the representational models in the introduction and interpreting their analysis with respect to the phenomenon in the conclusion (Abassian et al., 2020). The finding that students related the results from their analysis to the phenomenon of interest is promising, as this is a step that students have been found to skip when engaged in modeling (Crouch & Haines, 2007). Students also appeared to connect different representations to the same phenomenon in different ways. Specifically, students often transitioned from connecting formulas to the phenomenon in the introduction to connecting graphs to the phenomenon in the conclusion, indicating that students achieved one of the key skills of representational competence, which is the ability to use representations to describe phenomena and the ability to describe different representations of the same phenomena (Kozma & Russell, 2005).

In addition to differences in how they connected representations to the phenomenon, students' analyses of the symbolic and graphical representations differed. Their analysis of the formulas focused on describing how the different elements of the formulas mapped onto the phenomenon of interest, indicating that they were engaged in critical thinking about the meaning behind the formulas. However, they did not often engage in the same type of thinking as they analyzed graphical representations; instead, they focused primarily on the interpretation of graphical features without directly connecting to the phenomenon of interest. In addition, they primarily engaged in mathematical analysis via the graphical representations rather than via the formulaic representations. These practices may provide insight into the utility students see in the various representations and the type of thinking they engage in for each: specifically, that formulas served as a way to translate between mathematics and the phenomenon (as students engaged in simplifying and transforming), while the graphical representations were useful for the actual analysis and for communicating their findings. Students were more technical in their descriptions of the relationship between the phenomenon and formulas than in similar descriptions related to graphs, possibly indicating that they have more difficulty using mathematical language to connect phenomena to graphical representations compared to formulaic representations. This finding aligns with known challenges for students in developing skills for engaging in mathematical discourse (Schleppegrell, 2007) and connecting representations to phenomena more specifically (Ärlebäck et al., 2013). However, students did typically connect the outcomes from their analysis of individual graphs back to the phenomenon of interest, a connection not always seen in modeling research (Crouch & Haines, 2007; Stillman & Brown, 2021).

Students made comparisons between representations but were selective in how they did so. Specifically, students rarely compared formulas but did make some comparisons between graphs. In addition, students made connections between graphical representations and the formulas they represented visually but varied in whether they did this implicitly or explicitly. While drawing connections between representations is a component of representational competence, the ways in which students connected representations may be due to how they conceptualized the utility of the different representations. As described earlier, student pairs seemed to place more emphasis on the graphical representations for their analysis. Thus, students may see the symbolic, formulaic representations leading to the derivation of the visual, graphical representations that they then analyzed, hence building connections between the graphs and the related formulas but not between separate formulas. Similarly, as the graphical representations seemed to be students' preferred representation for analysis, they did not compare formulas but did compare graphs. This preference for graphical representations could be an artifact of the structure of the labs, where students generated graphical visualizations on the computer that they would not have been able to access otherwise.

Students successfully explained the limitations and appropriateness of representations for modeling specific phenomena and demonstrated an understanding of the ways that the representations were distinct from the phenomena they represent, both important components of representational competence (Kozma & Russell, 2005). Overall, students incorporated discussions of the limitations of the representations for modeling in their analysis sections, in which they discussed specific limitations of the formulas and graphs. The extent to which students discussed limitations aligned with the assignment descriptions and complexity of the models; this alignment is promising, as it indicates the utility for supporting certain stages of the modeling process via the structure of the WTL assignments. Students discussed the appropriateness of the models in the conclusion and related the appropriateness broadly to the phenomenon. That students discussed the appropriateness of the representational models is important, as it is an aspect of modeling on which students are known to place less emphasis (Crouch & Haines, 2007; Stillman & Brown, 2021) and indicates they are thinking critically about the applicability of the models. In addition, students discussed limitations with respect to both formulas and graphical representations but focused on the graphical representations when discussing the appropriateness of the models for the phenomenon. These findings parallel how students provided fewer connections to phenomena in their analysis, developing those connections instead in the conclusion. Students may have thus presented deviations from the phenomena more as limitations of singular representations, whereas in the conclusion, students were engaged in interpretation and thus contextualized how the representations best modeled the phenomena.

Conclusions and Implications

This exploratory study investigated how students used and analyzed representations in response to three WTL assignments. In their responses, students tended to follow a pattern of (1) introducing and explaining relevant formulas and connecting them to the phenomenon of interest, (2) transitioning from formulaic to graphical representations, (3) analyzing features of the graphical representations, and (4) synthesizing their findings from the graphical representations to make conclusions related to the phenomenon of interest. While students primarily analyzed the graphical representations, they successfully connected both formulaic and graphical representations to phenomena in ways that are necessary for mathematical modeling (e.g., mathematizing, validation) and demonstrated a certain level of competence with both types of representation. These results indicate that WTL may be a useful pedagogy for providing students with more opportunities to develop their representational competence and use various representations to model phenomena, which can in turn support their critical thinking as they make judgements and arguments about the phenomena based on their analysis of the representations.

Our findings also indicate the potential for WTL to elicit the specific skills necessary for representational competence and modeling. For example, students made connections between representations, although primarily between formulaic and graphical representations and, to a certain extent, between different graphical representations. These connections indicate the potential for WTL to support students as they translate between mathematical representations; however, an increased instructional focus on connecting formulas may be merited. In addition, all of the WTL assignments elicited students' consideration of the limitations and appropriateness of the representations for modeling specific phenomena, a part of the modeling process for which students are known to need support to achieve (Crouch & Haines, 2007; Stillman & Brown, 2021). The extent to which students made connections between representations and identified the limitations of representations for modeling phenomena across the three WTL assignments aligned with the assignment description and the complexity of the phenomena they were modeling. The differences between student responses to the three WTL assignments provide further evidence that the assignments can be tailored by instructors to support specific aspects of students' representational competence (e.g., translating between representations).

Further research is merited for examining how students' representational competence and modeling abilities are supported by the social interactions inherent in the implementation of WTL (i.e., working in pairs and going through a peer feedback and revision process), especially as social engagement has been found beneficial in other WTL studies (Finkenstaedt-Quinn et al., 2019; Finkenstaedt-Quinn, Polakowsi, et al., 2021; Gupte et al., 2021; Halim et al., 2018; Petterson et al., 2022; Watts et al., 2022). Furthermore, the WTL assignments that were the focus of this study did not have students going through a complete modeling process, in that students were provided with the models (Abassian et al., 2020). While the use of existing models limits the claims we can make about how WTL can support students in a more authentic modeling exercise, it does present a path for further exploration. Writing-to-learn assignments could be designed in which students are not given models to work with but instead need to determine or develop the appropriate models for themselves. Additionally, a series of WTL assignments implemented across a semester could be used to scaffold the modeling process, enabling students to gain autonomy in developing models as they progress through the semester.

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Appendix A

WTL Assignment Descriptions

Lab 1— Cancer

Imagine that you are a biomedical engineering consultant, and that a medical researcher has contacted you to give her insight on the development of cancer tumors. She is, in particular, interested in knowing what the model predicts for the behavior of the tumor in certain treatment regimes, and when different approximations to the model may be appropriate (and when they may be less so). Your writeup will be collaboratively produced by you and your partner, and both of you will submit the writeup paper. Note that you will need to include figures from the work that you did in the course of Parts A and B of the lab to produce a good writeup, and that you will need to include the equations and mathematical work underlies your conclusions.

The questions that the researcher has posed are the following:

- If a treatment reduces the rate of tumor growth, will that have a significant impact on the long-term outcome of the cancer?
- What is the predicted long-term behavior of the tumor, and would this be altered if the initial tumor size was changed, e.g., by a surgical intervention that removed most of the tumor?
- What type of behavior is predicted for the tumor by the simplified form of the Gompertz model? Is a simplified form of the Gompertz model adequate to predict the behavior of the tumor, and are there circumstances in which the simplification would be significantly better or worse?
- If the model is simplified by assuming a small tumor size, what can (and cannot) be determined from the resulting simplified model?

Lab 3 — Lasers

Your medical engineering consultant job was so successful that you have been hired by a company that is building lasers. Suppose that you have been asked to write a report to a scientifically minded prospective customer explaining the behavior of the lasers modeled in this lab. In your report you will want to address:

- What the models for the number of atoms in each energy state are, both without and with the energy pump, and how your eigenvalue analysis and solution plots illustrate the physical behavior of the system in either case.
- What additional effect the nonlinear system includes, what the equilibrium solutions of the system are, and what those suggest about the possible long-term behavior of the laser.
- How the stability of the different equilibrium solutions depends on the parameters in the problem, and what the linearization tells you about the stability and expected behavior or the nonlinear system.
- What the effect of a nonconstant parameter A is on the laser's output intensity, how this is similar to the phenomenon of resonance, and how the

characteristics of the output intensity in this case may or may not be desirable.

Lab 5 — Climate

Review the background description of the Lorenz system as a model of the motion of fluid between two layers, especially in the Prelab and Part A. Note that the functions x(t), y(t), and z(t) don't model the motion of individual particles. Instead, they describe the intensity of the motion of the particles in the fluid (x), the temperature difference between ascending and descending particles (y), and distortion from vertical motion of the particles (z). Then consider the lab report as described below. Next, we posit that you have had the revelation that the unifying theme in your varied lab writing career is your overwhelming love of mathematical modeling, and so have founded a consulting firm specializing in modeling and the analysis of mathematical models. A popular science reporter has contacted you to consider the impact of climate change, which has the effect of increasing the temperature at the Earth's surface, on weather forecasting. You are writing a report in response to her request using a simple model (the Lorenz system) that captures some of the behavior of the atmosphere while avoiding the need to explain a far more complex model. In your report you will want to address the questions:

- How your linear analysis of the system at the different critical points allows you to predict its behavior when $r < 24.7368$ ..., and how this is different when $r > 24.7368$ …
- How the case $r > 24.7368$ ... exhibits sensitivity to initial conditions. Use your work from Part B, Exercise 1 to demonstrate on this, and reflect on what it means for weather forecasting.
- How nonlinear systems may exhibit behavior that result in long-term prediction of their behaviors being difficult to impossible, and the implications of this for weather forecasting.

Appendix B

Coding Scheme with Definitions and Exemplars

Code	Definition	Exemplar
Use of Graphs/Figures	Student includes a graph or figure.	*"Like the impact of the rate, the impact of the K value was also demonstrated in Figure 1."*
Presentation of Graphs/Figures	Student references specifically back to the graphs/figures and presents them clearly.	*"As demonstrated in Figure 1, specifically by the comparison of the blue line to the maize line or the red line to the green line…"*
Use of Formulas/Equations	Student includes formulas/calculations.	*"The Gompertz equation described above is defined for this paper as:* $y \ln(\ )\ {}_{dt}{}^{dy} = -r_y K$ *"*
Presentation of Formulas/Equations	Student references back to calculations/formulas and presents them clearly.	*"In particular, we modified the r and K values, which are the positive constants in the Gompertz equation"*
Motivation	Student explains what the representation they are using is modeling.	*"the use of the Gompertz equation as a model for the behavior of cancer cells over a period of time"*
Explaining Assumptions	Student explains various assumptions associated with the equation and/or student explains what specific variables are set to.	*"It was assumed that K was equal to 10, and r, the rate of cell growth, was equal to 0.1. "*
Defining Variables	Student defines various values on the graph or values in the equation or mathematical terms.	*"where K and r are positive constants and the function y(t) gives the volume of the tumor at a time t."*
Stating Why Representation is Appropriate	Student generates the graph or equation and explains why it is appropriate to use.	*"Practical applications of this can be for laser surgery, where the intensity may need to be adjusted depending on the precision of the procedure, based on the type and thickness of tissue that is being cut."*
Limitations	Student describes the limitations of the model by explaining when it is not appropriate to use.	*"Our model will show us how large the tumor will grow, but it will be unable to accurately show us how quickly this growth will occur due to large amounts of error at small amounts of time."*
Discusses Assignment	Student references the assign-ment or the questions they were asked to answer.	*"One of the questions posed to us asks about how changing the rate of growth changes the total growth."*
Creating Models Using Technology	Student references the technology they used to create the representation they present.	*"Using MatLab we are able to analyze the behavior of these functions over a long period of time."*
Overall Purpose	Student explains overall purpose of the assignment at hand.	*"This paper serves to demonstrate the use of the Gompertz equation as a model for the*

Code	Definition	Exemplar
		behavior of cancer cells"
Relating Back to Context	Student gives context to the mathematical concepts/results they are explaining.	*"This is useful when considering this equation as a model of cancer cells because it shows us that the long term growth of the tumor is left completely unaffected by its initial size before or after a surgical intervention"*
Analyzing Features of a Representation	Student use words to describe specific behaviors of the representation.	*"This can be seen in the graph as well, because both solutions with a K value "k1=0.1" converge at 0.1 while the solutions with "k2 = 0.2" converge at 0.2."*
Comparing Different Representations	Student discusses the relationship between multiple representations.	*"Our simplified model also describes this same behavior but not quite the same as the Gompertz equation."*
Drawing Conclusions Based on Representations	Student draws broader conclusions/makes inferences based on the features of their representations.	*"As you can see in the graph below, modifying the r values caused the short-term behavior to change, while modifying the K values caused the long-term behavior to change. When the r value was increased, the graph reached its equilibrium point faster compared to the original, smaller r value."*