Abstract Algebra and the Conversation of Humankind

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Abstract: Peer review is especially difficult to facilitate in advanced mathematical writing. Typically, only someone with an appropriate level of disciplinary knowledge can understand the workings of a mathematical proof, for example, let alone provide useful feedback to a novice proof-writer. This presents a challenge to writing programs and writing centers charged with supporting writing throughout the curriculum. In this article, we discuss our efforts to support student proof-writing in an advanced abstract algebra course, in which students are expected to write their own sophisticated proofs of challenging mathematical propositions. Building primarily on the work of Ken Bruffee, we assert that math proofs are a form of normal discourse. Bruffee (1984) contends that collaborative learning is an especially good way for students to practice normal discourse with an audience of knowledgeable peers. In such an arrangement, the student, teacher, and peer reviewer each make different contributions to the learning experience. The peer reviewer, in our case, is a trained undergraduate writing consultant. Our analysis of teaching and learning artifacts, formal and informal student evaluations of the course, and transcripts of a student focus group, leads us to conclude that the collaboration has two observable outcomes: first, we get a higher percentage of student-written proofs that demonstrate an understanding of threshold concepts in abstract algebra; and second, students learn to communicate better and become members of the mathematical discourse community. We contend that these two are recursive and cannot be separated.

> Our task must involve engaging students in conversation among themselves at as many points in both the writing and the reading process as possible, and that we should contrive to ensure that students' conversation about what they read and write is similar in as many ways as possible to the way we would like them eventually to read and write. The way they talk with each other determines the way they will think and the way they will write.
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> —Bruffee, 1984, p. 642).

In a 2014 *WAC Journal* article, Bryant, Lape and Schaefer refer to the mathematics department as "the final frontier" (p. 92) for WAC/WID programs. While these departments have been leaders in the use of writing-to-learn pedagogies, it seems that WAC programs have been able to offer them little assistance or support in the teaching of disciplinary mathematical writing. Proof-writing, arguably

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the foundation of mathematical discourse, seems to defy the practice of process writing as it is taught in composition courses. The *sine qua non* of a mathematical proof is a series of arguments that are objectively verifiable. If that series is wrong in any way, a writer simply does not have a proof. The kinds of invention strategies taught in composition classes don't seem of much use to a math student trying to write a proof. Conversely, composition specialists would be hard pressed to recognize as rhetorical "invention" the kinds of brainstorming and other pre-writing activities mathematicians do. Drafting, for an undergraduate proof-writer, consists primarily of applying the proper proof techniques to the problem at hand. Once students have proved their propositions, it can be difficult for them to imagine a place for revision. As one of our students puts it, "you could be Shakespeare with the pen, but if the math isn't right, it doesn't matter."

Peer review, in particular, is especially difficult to facilitate in mathematical writing. As Bryant, Lape and Schaefer put it, "Peer review runs the risk of becoming an empty exercise in which unknowledgeable students provide equally unknowledgeable students with faulty feedback" (2014, p. 92). Yet, if the instructor provides the only feedback on the students' proof-writing—feedback which typically consists of correcting errors—"there is nothing left for the writer to 're-see' and to revise" (p. 92). What, then, can a WAC program offer instructors of advanced mathematics, or their students who are learning to write the discourse of their discipline?

In this article, we begin with the same assumption as Bryant, Lape and Schaefer (2014), that proof-writing is a form of disciplinary discourse that challenges our traditional assumptions about how best to support writing in a specific discipline. Bryant, Lape, and Schaefer focus their support efforts on a gateway course in theoretical mathematics, in which students first learn the fundamental techniques of proof-writing. In this article, we discuss our efforts to support student proof-writing in a next-level Abstract Algebra course, in which students are expected to write their own sophisticated proofs of more challenging mathematical propositions than those typically encountered in a gateway course.

There are, in this course, threshold concepts that are essential to any students who wishes to continue their mathematical studies in graduate school. A threshold concept, according to Land, Cousin, Meyer, and Davies (2005), "represents a transformed way of understanding, or interpreting, or viewing something without which the learner cannot progress" (p. 53). Readers of *Across the Disciplines* are likely familiar with the work of Linda Adler-Kassner and Elizabeth Wardle on threshold concepts in writing studies (see, for example, *Naming What We Know: Threshold Concepts of Writing Studies*, 2015). For our purposes here, we find it especially important to note that threshold concepts "also entail a shift in learner subjectivity and an extended use of discourse" (Land et al., 2005, p. 53). It is our contention that such shifts in learner subjectivity are inextricably bound with the extended use of discourse, and that the process of learning to use specific kinds of discourse is, in fact, integral to the learning of threshold concepts in mathematical proof writing.

Building primarily on the work of Ken Bruffee, we assert that math proofs are a form of normal discourse. In order to facilitate a deeper engagement with proof-writing as discourse, we assign a trained undergraduate writing associate to each participating section of Abstract Algebra. Bruffee contends that collaborative learning is an especially good way for students to practice normal discourse with an audience of knowledgeable peers. In such an arrangement, the student, teacher, and writing associate each make different contributions to the learning experience.

As of this writing, we have assigned writing associates to Abstract Algebra for four years. Each iteration of the course has been offered by a different instructor, each teaching two sections of the course simultaneously. Nearly 150 students and eight undergraduate writing associates have been part of our collaboration. Our observations come from teaching and learning artifacts collected during three of these four years, including assignment prompts, student-authored proofs, reflection essays written by students, and the writing associates' conference notes. Additional information comes from formal and informal student evaluations of the course, and from a focus group conducted with a cohort of students from the first iteration of the course to be supported by a writing associate.

Background

A college-wide writing program at our institution supports teaching and learning with writing throughout the curriculum by directly supporting faculty in the design of their writing courses, and by hiring, training, and supervising a staff of undergraduate peer writing tutors, known locally as writing associates. As part of our common course of study, students must complete at least one writing-intensive course in their own major. The purpose of this WID requirement is to ensure that all students receive at least some guided practice in the disciplinary writing of their field. Liz, who has extensive experience teaching theoretical mathematics to undergraduates, was instrumental in getting our Abstract Algebra course recognized as writing-intensive. In spring 2015, Liz approached Christian, in his role as coordinator of the college writing program, about assigning writing associates to her two sections of Abstract Algebra. Tom teaches many of the same mathematics courses as Liz and has also worked closely with our institution's college writing program for more than twenty years. For nearly a decade, Tom had been helping the writing program interview prospective writing associates. Based on recommendations from Liz, we worked together to select two associates who not only knew the mathematics, but could also facilitate discussions with fellow students about the rhetorical concerns of effective proof-writing. Sam was a student in that first class and also a writing associate for other courses outside of mathematics. She went on to be the writing associate for two future iterations of Abstract Algebra before graduating and moving on to pursue a PhD in mathematics.

Like many college mathematics departments, our institution offers a gateway course in theoretical mathematics, which we call Transition to Theoretical Mathematics (hereafter shortened to "Theoretical Math"). Theoretical Math provides our math majors "an introduction to the concepts and techniques that permeate advanced mathematics," with a "special emphasis on developing students' facility for reading and writing mathematical proofs" (Lafayette College, 2018a). In this course, students learn canonical techniques for structuring and supporting mathematical propositions: proof by induction, proof by contradiction, counting arguments, set theoretic techniques, etc. Professors and students alike say that in Theoretical Math, "you learn what a proof *is.* You begin *writing* proofs in Abstract Algebra." What, exactly, our students and colleagues mean by this requires some elaboration.

Students do indeed write proofs in Theoretical Math. Most find the endeavor both novel and difficult. As Kaitlyn, one of Liz's students observed, "we've been writing English since kindergarten, but I started math proof writing last year," and as Ben, another one of Liz's students commented, "we have an entire course that's built around learning how to write math, and we all struggle with it." The assignments in Theoretical Math call on students to construct proofs in particular ways with particular features. For example, they learn that to prove two sets are equal, the expectation is that they will show that each set is contained in the other, or that to prove two sets are the same size, they should produce a bijection (a one-to-one correspondence between the elements in each set). The proofs in Theoretical Math tend to be short and highly structured, and provide frequent opportunities for feedback from the professor. The feedback is primarily concerned with the logic of proofs, a primary course objective, and not with style. As Ben notes, "you should be able to *structure* a proof by the end of [Theoretical Math]" (emphasis added). To be clear, Liz has always given students in Theoretical Math feedback on style, but it tends not to be a primary determinant of students' grades in the course.

It is important for us to note that even at the "gateway" level of Theoretical Math, learning to write in a discipline specific way is inextricably bound with learning to *think* in a discipline specific way. "You can't understand the subject unless you prove things," Tom likes to say, "and in the process of proving things, you're really deepening your understanding." Likewise, Liz tells her Theoretical Math students, "you can't prove something until you truly believe it's true." Oftentimes, however, a proof-writer doesn't start out believing the thing she's going to prove. Or, she might believe it, but have absolutely no idea *why* it's true. Theoretical Math students are learning to work with the tools of proof-writing. Once they develop competence with these tools, they are then able to begin proving things with intention. Or, as Tom likes to say, in Theoretical Math, students learn to prove. In Abstract Algebra, they begin proving to learn. Correctness is a necessary but not sufficient condition (language that mathematicians will often use in discussing elements of a proof) for writing to a wider mathematical audience. Writers at this level must also consider how to persuade a reader that their proof is complete and logical. Well-written proofs are organized not necessarily according to the writer's thought process, but in a way that makes the most logical sense for a reader, and it is here that we found the opportunity for the writing program to support Abstract Algebra.

Normal Discourse

A well-written math proof, particularly at the undergraduate level, is an example of "normal discourse," the discourse commonly used in a knowledge community, which follows set conventions and helps members of the community talk about and affirm what they know. Here's how Ken Bruffee puts it: "normal discourse is pointed; it is explanatory and argumentative. Its purpose is to justify belief to the satisfaction of other people within the author's community of knowledgeable peers" (Bruffee, 1984, p.643). The purpose of a mathematical proof is precisely to justify the writer's belief in a mathematical proposition to the satisfaction of others within the mathematical community. An especially salient affirmation of mathematical proofs as normal discourse comes from Mingus and Grassl (2010), who write, "Proof not only provides the foundation upon which mathematical ideas are built, but also the way for each generation to learn about and extend what has already been accomplished" (p. 438). Correctness, as we have said, is necessary, but not sufficient in normal mathematical discourse.

In the following problem, for example, the student has to justify her belief that the union of two subgroups with specific properties is not itself a subgroup: "Suppose H and K are subgroups of the group G. Suppose that H is not contained in K and K is not contained in H. Prove that $H \cup K$ is not a subgroup of G." That the belief is true is critically important, but it is not in itself sufficient. The student in this case knows that the belief (the union of these two subgroups is not itself a subgroup) is true, and may have an intuitive feel for why, but her argument is not convincing unless she can demonstrate explicitly that that $H \cup K$ fails to satisfy at least one of the defining properties of a subgroup in all cases when H and K are as described above. It is the construction of such a mathematical argument, one that convinces other mathematically sophisticated readers, that makes this normal discourse.

Collaborative Learning

Bruffee's (1984) work emphasizes the learning value of working with peers, rather than teachers alone. The power of collaborative learning, including peer writing conferences, is precisely that it gives students practice addressing "a community of status equals: peers" (p. 643). Bruffee theorizes that the peer writing conference is a powerful opportunity to practice normal discourse because of the pooled resources of the three main players. In language loosely paraphrased from Bruffee, we summarize the contributions of each below:

1. The student, or tutee, brings knowledge of the subject to be written about and knowledge of the assignment.
2. The tutor, in our case, a writing associate, brings not only knowledge of conventions of discourse (not at the professional level, but typically at a more advanced level than the student), but also, by way of his or her tutor training and peer status as a fellow undergraduate, sensitivity to the needs and feelings of peers.
3. The teacher brings expert-level knowledge of the discourse community she or he represents, and structures the assignment in such a way that students must follow the conventions of that community.

In the following sections, we discuss what we believe the teachers (Tom and Liz), the writing associates (Sam and her colleagues) and the students of Abstract Algebra each contribute to the collaborative learning in the course at our institution.

The Student

Bruffee (1984) contends that students (tutees, in his words) contribute to the pooled resources of collaborative learning “knowledge of the subject to be written about and knowledge of the assignment” (p. 644). “You have to understand it,” says Katelyn, a student in Liz’s class, “You have to know what’s going on, how each theorem leads to what you want before you write it.” From that point forward, though, Katelyn says, “it’s like a give and take…kind of back and forth.” Like many students working on an academic writing assignment, Katelyn thinks she knows what she wants to say from the outset. But it’s not until she attempts to explain it to another person, until she “hash[es] it out,” that she truly appreciates how all the steps of her proof flow together toward a conclusion, that she knows for sure. Carly, another student in Liz’s class (who also happened to have been a writing associate), described the knowledge she brought to the conference as “a degree kind of thing.” She explains it this way: “if someone asked you to write an essay on salamanders, and you didn’t know what a salamander was, you’d have a pretty hard time, right?” But if you have at least some understanding of the subject, Carly says, then “it’s really a problem of formulating it.”

Sam herself, when she was a student in Liz’s class, had this to say:

> A lot of my conferences were mostly me bouncing ideas off of [the writing associate]. And since he knew what the problem was asking, I could say what I was thinking but not actually say it, because I didn’t know how to say it. And he [the writing associate] would understand what I was trying to say, so he would help me figure out how to actually prove it…how to say it in my proof.

Sam, Katelyn, and Carly each recognize that they brought to their writing conferences an understanding of the problem and at least the bare bones of a solution. Without that basic understanding, the exchange between tutee and tutor simply could not take place.

It’s important for us to note here that the writing associate is not a math tutor. When she worked as a writing associate for Tom’s sections of Abstract Algebra (as well as two other sections not analyzed for this article), Sam and her fellow writing associates began each conference at a point where the student had already worked out most, if not all, of a viable proof. Liz’s student Carly told us in a focus group that she didn’t quite know how to operate with the writing associate at first. The meetings were “a little challenging,” she says, “because it was like, OK, are we gonna talk about the math or not?” The students soon realized, however, that they were talking with someone who had a deep understanding of the course content, which is not always the case for a writing associate. They knew, in other words, that they were conversing with a knowledgeable peer.

The Writing Associate (Tutor)

The associates in our writing program receive extensive training and professional development in writing conference pedagogy. Like most writing programs and writing centers, we embrace a facilitative conference pedagogy, encouraging our associates to pose thought-provoking questions to encourage reflection and critical thinking. An outline of the curriculum we use to train our writing associates is provided in an appendix to this article.

Heeding Mike Rose's (1980) exhortation to adopt a heuristic, rather than a formulaic, approach to tutoring (p. 391-2), we turn to Joseph Harris's *Rewriting: How to Do Things with Texts* (2017), and in particular, to his chapter on Revising. Harris suggests that the four main "moves" outlined in his book (coming-to-terms, forwarding, revising, performing a take) can also be used reflexively, as a way to make visible the practice of revision. He turns each of the moves into a broad question, or set of questions that students can ask of their own writing-in-progress:

- *What's your project?* What do you want to accomplish in this essay? (coming to terms)
- *What works?* How can you build on the strengths of your draft? (forwarding)
- *What else might be said?* How might you acknowledge other views and possibilities? (countering)
- *What's next?* What are the implications of what you have to say? (taking an approach) (p. 100, emphasis in original)

Christian has the writing associates read this chapter from *Rewriting* and encourages them to use Harris's four questions as a heuristic for approaching student drafts. In the sections that follow, Sam describes how she uses each of these questions in her conferences with Abstract Algebra students to get them thinking critically about their proofs. The aim of this approach, as Harris puts it, is to return to one's draft "in order to make your thinking…more nuanced, precise, suggestive, and interesting" (p.99).

What's Your Project?

The first step in reading a proof is understanding what is being proven, or what the end result should be. This is the essence of the "what's your project?" question. By asking students what the end goal should be, we encourage them to think harder about what it is they are really trying to prove. In most cases, the answer is simple: they're trying to prove the theorem they've been given. However, while writing a proof, many students lose sight of the end goal. For example, in the following assignment, there are lemmas—smaller, already proven propositions (a.k.a. helping theorems)—that should be included in the proof in order to prove the original theorem.

> We plan to prove the following theorem: Let G be a group with M and N normal subgroups of G. Suppose that both G/M and G/N are abelian. Then, $G/(M \cap N)$ is abelian.
>
> Step 1): Prove the following lemma: Let G be a group and let $N \lhd G$. Then G/N is abelian if and only if for every $x, y \in G$, we have $[x, y] = xyx^{-1}y^{-1} \in N$.
>
> Step 2): Prove the following lemma: Let G be a group with both M and N normal subgroups of G. Then $(M \cap N) \lhd G$.

Step 3): Prove the theorem.

Note that there are three steps: prove each of the two lemmas and then prove the theorem. Students can easily get lost in such a problem, and their proofs are often aimless and fragmented, with no clear direction. In particular, students lose sight of the end goal: proving that $G/(M \cap N)$ is abelian. Instead, they prove the three statements separately, without a sense of cohesion. When a peer—especially one trained in writing conference pedagogy—asks students what their project is, she is encouraging them to think about why these three statements are together. What is the relationship between these smaller results, and how can they be combined into a larger result? Then, students can bring the steps together, and use them as building blocks towards showing $G/(M \cap N)$ is abelian, thus resulting in a clearer, more precise proof of the theorem.

To put it another way, the writing tutor in these conferences is helping the student to absorb a threshold concept in advanced mathematics. They are facilitating what Anna Sfard (1991) referred to as "reification," or the ability to think of abstract notions "operationally as process and structurally as objects" (Breen & Oshea, 2016). The lemmas, in this proof, become objects, reified concepts that are in turn used to prove a larger, more complex proof. Sfard describes such reification as nothing less than "an ontological shift, a sudden ability to see something familiar in a new light" (p.19).

What Works?

When a writing associate asks her student "What works?" she's trying to help the student identify the strengths of the draft and build on them. When the draft is a mathematical proof, students might answer from two different perspectives: from a mathematical perspective or a rhetorical perspective. While there are definitive overlaps between these points of view, students often ignore the rhetorical. Instead, they focus on the correctness of the proof. If the writing associate is not careful to remain facilitative and also maintain focus on the rhetoric of the student's proof, the question "What works?" could quickly be reduced to "Is the math correct?" However, when a proof is viewed as rhetoric, the question "What works?" now becomes something more like, "Is this a convincing argument?" Can a reader believe that the next statement follows from the previous?

What Else Might Be Said?

Harris uses the question "What else might be said?" as a way of encouraging writers to acknowledge viewpoints and consider approaches other than those reflected in the current draft. In mathematical discourse, there are often multiple ways of arriving at the same conclusion. Some paths are more convincing than others. By asking students "What else might be said?" or "How else might you prove this?" the writing associate is asking students to acknowledge that there is more than one way to prove a proposition, and in turn, to think carefully about which approach will be most convincing for an audience.

What's Next?

For many students, a statement and its proof are a small result, one statement to prove and then move on. Students will rarely stop and consider the implications of their proof. But when a writing associate asks, "What's next?" it encourages the student to think beyond the proof in front of her. For example, consider the following exercise: "Let H be a subgroup of G with $|G:H|=4$. Suppose that $g \in G$ with $|g|=5$. Prove that $g \in H$." If the student has written a valid proof in response to this problem, the writing associate can help her extend her thinking by asking "What's next?" Does your proof extend to different subgroups of higher index? Is there something special about the numbers 4 and 5, or do you think your proof can be generalized to a much broader result?

In fact, one student alluded to the "What's next" question in a reflection assignment. In this assignment, Tom asked the students to

> Find your favorite homework problem from the problems I assigned from the groups section of the course and write it up as if it appeared in the course text. If you are proving a result, state the theorem in the style of the text and then prove it in that same style. If the problem is a computation, write it up in the format for examples the text uses.
>
> Now explain to me why you picked that particular problem. What is it about the problem you like, either by itself, in the context of the course, or some combination of the two? This explanation should be a minimum of 100 words.

One student gave the following explanation of why the previously given example was his favorite homework problem:

> This problem is my personal favorite of all of the problems we have done thus far, as while it is seemingly simple, the result is actually interesting. Just given that the index of a subgroup H in G is four and that the order of an element $g \in G$ is five, you know that $g \in H$ as well. While simple on the outset, this can be extended to any prime larger than the index, and you can even change the size of the index and it still holds true. This extension beyond the scope of the problem is one of the more interesting parts of the problem for me.

This student touches on one key point in his explanation of why he chose this particular problem. He says "this can be extended to any prime larger than the index." This is a great answer to the question "What's next?" because he is already thinking of this problem in a broader context.

The Teacher

The types of writing valued in learned communities do not come naturally to students; teachers must provide guided practice to lead students to these forms of writing. When students learn collaboratively, their conversations are structured by the tasks their teachers have designed. Assignment design, in other words, is crucial. "Students are especially likely to be able to master [a new] discourse collaboratively," argues Bruffee (1984), "if their conversation is structured indirectly by the task or problem that a member of that new community (the teacher) has judiciously designed" (p. 644). In the context of Abstract Algebra, many problems that would ordinarily work quite well for teaching the mathematical concepts involved in, say, set theory, are limited in their value for teaching mathematical discourse. Consider the following example from the first set of problems that both Tom and Liz gave their students to work on with their writing associates:

> Suppose S is a finite set with a binary operation $*$, which is both closed and associative. Suppose also that $*$ obeys both a left cancellation law and a right cancellation law. That is:
>
> Left Cancellation: For every x, y, z $\in$ S, $x * y = x * z$ implies that $y = z$.
>
> Right Cancellation: For every x, y, z $\in$ S, $x * z = y * z$ implies that $x = y$.
>
> Prove that S is a group under the operation $*$.

This problem presents a proposition to defend, and in fact goes so far as to suggest that the proposition is indeed true. What's left to the student is to marshal the mathematical evidence necessary to prove it.

As it turns out, however, the problem asks students to do multiple things which are not at all obvious, especially to an undergraduate student. When it came time to meet with the writing associate, students were still working out their preliminary solutions to this problem. The students found it difficult to practice discourse with the writing associate when they were still working out the basics of their solutions. There was not enough time for them to talk through their solutions with the writing associate and reflect on and revise their proofs. Bruffee (1984) asserts that "what students do when working collaboratively on their writing is not write or edit or, least of all, read proof. What they do is *converse*" (p. 645, emphasis added). If an otherwise perfectly teachable problem does not allow space for such conversation, it will not be a good candidate for collaborative learning.

As Bruffee (1984) himself acknowledges, such conversations among students can and do break down (or, in the case of the previous problem, fail to get started). "It can proceed again," however, "if the person responsible for providing the missing element, usually but not always the teacher, is flexible enough to adjust his or her contribution accordingly" (p. 644). Liz and Tom made one such adjustment to their assignments by intentionally wording problems in such a way that encourages metacognition and, ultimately, the use of metadiscourse. An illustration of this comes from the second problem in this first homework set:

> Let S be the set of functions from R to R of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. Define a binary operation on this set given by $fg(x) = (f \circ g)(x) = f(g(x))$. So the binary operation is function composition. Prove that the set S is a group under this binary operation. (You may assume that function composition is associative.)
>
> Is S an Abelian group? Why or why not?

Several things about this problem encourage students to think about what will be convincing to a mathematically sophisticated reader. The first is the simple question, "Why or why not?" The wording is deliberate, but it is not mathematical language, per se. With these four simple words, Tom is beginning to make explicit the rhetorical context of the problem by assigning his students a role in that context. To use John Bean's (2011) language from *Engaging Ideas*, Tom is helping his students "understand the kind of change they hope to bring about in their audience's view of the subject matter" (p. 99). In his next set of problems, Tom invites his students to think even more deeply about their audience and the role he expected them to play in their proof writing.

The problem above had an additional challenge for students: "Suppose we omit the condition $a \neq 0$. Then S is no longer a group. Discuss exactly which group properties hold, which fail, and justify your claims in that regard." Tom could easily have given them the instruction, "either prove that these properties hold, or produce a counterexample," which would be the typical language of a homework assignment in Abstract Algebra. But by challenging students to "discuss which properties hold, which fail, and justify [their] claims," Tom is explicitly tasking them with considering how to convince the audience. We might also note that this problem contains language such as "discuss" and "justify your position," which is far closer to a traditional writing prompt than one might ordinarily encounter in Abstract Algebra.

A third, closely related adjustment came in giving the students specific guidance about who, exactly, their audience is. The following notes appeared at the end of Tom's second problem set of the semester:

> Notes: Your target audience for all of the WA [writing associate] homework (and homework in general) is the same audience as readers of your text. So your reader is not a mathematician, but has some mathematical sophistication. You don't have to define standard mathematical symbols, but if you look at your text, you will see most of the proofs and explanations are not heavy on use of symbols or notation. You should go easy as well. Something like
>
> $\forall x \in S, \exists y \in T$ such that $xy \in H \Rightarrow x = y$
>
> is not in the style of the text. Your audience would much prefer to read:
>
> Let x be an arbitrary element of S. Then there is an element y in T so that if xy is in H, then x = y.
>
> In short, please behave as if you are writing a section of the text for the audience described above.

By specifying an audience of mathematically sophisticated textbook readers with clear stylistic preferences, Tom is enabling his students to think about what their proofs might do for a reader. As Bean (2011) would describe it, Tom is inviting his students to write "from a position of power" to an audience who knows less about the topic than the writer does. It is not enough for Tom's students to prove or disprove the propositions in this assignment: they must go further and make their proofs clear to readers of an advanced college algebra text.

Here is the introduction to one of the problems in this assignment:

> I am asking you to prove the following theorem. I will provide you with an outline of the proof. Your job is to convert the outline into a coherent proof with all steps justified. You may use results we have proved either in class or in the homework in this proof. Also, the following result (which you may have seen in 290) may be quoted. Let gcd(a,b) = 1. If a divides n and b divides n, then ab divides n.

Notice how much more is going on in this problem, over and above the mathematics. First, Tom is describing something of the body of common knowledge of the readers. Students often struggle with how much or how little to include when writing to an audience with some level of sophistication. As Aditi, one of Tom's students observed, writing for an audience of textbook readers was a very new challenge for her.

> I am used to justifying every step of a proof and providing reasoning for why each statement is true. However, the style of the textbook presumes some prior knowledge of the details of the proof, so each step is not explicitly explained. This caused some trouble for me at first because I was unaccustomed to writing a proof in that style, but it does make sense that a textbook, which is to be read primarily by people who know or are learning the material, does not necessarily have to provide justification for every step because the readers should have the prior knowledge to understand why each step is true.

What makes the problems we've discussed here different from the more traditional proof-writing that students still do in Tom and Liz's classes is their intent to build in room for students to think about their rhetorical choices. These assignments create space for drafting and revising. Once the student has written a draft of these proofs, they meet with the writing associate to talk through their solutions. The focus is not directly on the mathematics, but the logic of the proof (again, correctness is necessary, but not sufficient). To be sure, the student must have the mathematics correct, and if there are flaws in the proof, they must be addressed first. But the real purpose of the writing associate conference is for students to have a conversation with a knowledgeable, albeit non-expert peer, a conversation that is focused on how the student can communicate their mathematical thinking to a wider audience.

Outcomes

Upon successful completion of Abstract Algebra, we expect students to be able to work with algebraic objects and to construct complete and correct proofs of some elementary results concerning those objects. In any given semester, we find that a majority of students are indeed able to do these things. We believe that two additional things happen as a result of our approach to teaching Abstract Algebra as a writing class and collaborating with the writing program:

1. We get a higher percentage of proofs that demonstrate a deep and clear understanding of the underlying mathematics.
2. Students learn to communicate better and become members of the discourse community.

We contend that 1 and 2 cannot be separated. They are self-reinforcing. The act of trying to communicate a student's ideas to another forces a self-reexamination of those ideas and leads to both reorganization and revision of the mathematical arguments, which simultaneously leads to a deeper understanding of the mathematics and makes it possible for the student to communicate those ideas more effectively. The students, moreover, seem more invested in the idea of writing to a genuine audience of mathematically savvy peers. They pay more attention to the quality of their writing, and that attention appears to carry over into other writing they do in mathematics.

Tom was surprised to see that this attention even seemed to carry over to his exams. One of the last problems on Tom's third in-class exam of the semester was the following: "Let H and K be subgroups of G and suppose that there are elements $a, b \in G$ such that $aH \subseteq bK$. Prove that $H \subseteq K$." The objects in the question were familiar to the students, and the proof is not long or complex, but it is a bit awkward. Moreover, there are multiple paths a student might take to a solution. When Tom showed the problem to one of his colleagues before the exam (a veteran Abstract Algebra teacher), he stumbled a bit before producing a solution. Tom was expecting to see numerous false starts and convoluted proofs in the student write ups. In fact, such write ups were rare. Students took several different paths to the solution, but almost all of them were direct paths, and they were delineated clearly by the student author. These observations led Tom to make one critically important inference: there was simply no way all of these proofs were first drafts. Here's what he thinks happened: The majority of students worked out their arguments on scratch paper, and *then* transformed those arguments into a coherent proof.

For students to draft and then revise a proof during a timed, in-class exam strikes Tom as remarkable. To the best of his recollection, Tom had rarely seen prose on an in-class exam that so effectively considered the needs of the reader. The quality of most of their solutions, moreover, was at a level he would have expected to see (and did see) on their later weekly problem sets, which naturally lend themselves not only to working and re-working the proofs, but to drafting and revising the prose as well. These students appear to have incorporated, with the needs of a mathematically savvy readership in mind, revision as a natural part of all mathematical exposition. It was at this point Tom realized something special had happened as a result of the collaborative discourse process described previously.

How do students perceive this (for them) new process for producing mathematical discourse? From their perspective, the conference with the writing associate is the most noticeable change, and their thoughts regarding it are enlightening. At the conclusion of her Fall 2015 course, Liz asked her students to comment anonymously on their experience working with a writing associate in Abstract Algebra. Here are some of the more illuminating responses.

One student told Liz that the writing associate "often helped to break down logical steps in how to prove a desired 'thing,' which was helpful because often the problem was made harder by myself just thinking about it too much." Other comments dealt with conciseness, as in "filtering out unnecessary info." Conversely, other students say they learned to offer more complete explanations, recognizing the need to "explain every statement I claim." Liz's students also commented on features of their writing that they tended to identify as stylistic. One student believes working with the writing associate "made me think more about writing style choices, rather than purely mathematical content." Others referred to structure and organization, and even the process of revision.

Such comments strike Christian and Tom as examples of students learning to transform writer-based prose into reader-based prose, a concept first described by Linda Flower (1979). Untransformed writer-based prose is characterized by an egocentric, narrative structure; it is the story of how students arrived at a conclusion and is often presented either chronologically ("first, I did X; next, I did Y") or in the form of a survey of ideas. The problem with such prose, as Flower puts it, is that "the reader is forced to do most of the thinking, sorting the wheat from the chaff and drawing ideas out of details" (p. 25). Many student-written mathematical proofs do just this: they demonstrate the process by which the student arrived at an answer, but the resulting prose is of limited use to and typically fails to convince a reader of its mathematical coherence and completeness.

Flower (1979) believes that writer-based prose serves an important function for authors. "Because dealing with one's material is a formidable enough task in itself," she writes, "a writer may allow himself to ignore the additional problem of accommodating a reader" (p. 27). It is an "economical strategy" for writers to cope with information—especially new and complex information such as that discovered in the course of writing a proof in Abstract Algebra. With this understanding in mind, Flower urges teachers to allow students space in which to harness the cognitive power of writer-based prose and offer support in transforming it into prose that is more reader-based. Christian believes it is not at all coincidental that the writing associates are familiar with Flower's work, and with that background, we frequently discuss the need to recognize writer-based prose and help facilitate its transformation into reader-based prose. Ben, a student in Liz's class, speaks to the writing associate's role in his transformational process: "A lot of times you work, you work, you work, and then you find this one little piece that actually drives the proof. [But] that's not necessarily the best way to describe it." Jason, Ben's writing associate, helped him see that:

> A trap I usually fall into is that I jump straight to the details of how I figured it out, and then I explain all the necessity to get there. But [the writing associate] did a really nice job of [explaining to me], "there is a linearity in what you're trying to explain, there is a story to tell through this proof, so start at the top, talk about it, and flow through it, and don't just jump to the finish." Jason said that to me a lot.

As Flower (1979) writes, "This transformation process may take place regularly when a writer is trying to express complicated information which is not yet fully conceptualized. Although much of this mental work normally precedes actual writing, a first draft may simply reflect the writer's current place in the process. When this happens, rewriting and editing are vital operations" (p.28) The approach we've written about in this article aims not only to provide space for such rewriting and editing, but to harness their power as tools for learning mathematics. By helping students to revise and edit their proofs, the writing associate is helping them to realize more fully and concretely what they are learning in Abstract Algebra, while simultaneously helping them to enhance their skills of mathematical discourse. The conventions of a discourse community are bound up in the ways of thinking that community values; the two are, in essence, inseparable. Therefore, if the students are getting better at mathematical writing, they are inevitably getting better at mathematical thinking.

Having been able to write proofs clearly and understand them at a deeper level prepares students for what's ahead for them as mathematicians. They've learned more than how to write a better proof. The process of revising gives them a deeper understanding of what they just proved and prepares them for what's ahead: the unknown. Cedric, a student in one of Tom's sections, made this observation in his reflection assignment:

> I love this problem because it is a perfect example of how we explore and prove something we don't currently know. And personally, I think this problem implies the central conception of math. In this problem, [the] first isomorphism theorem is the floor we are standing on, and the second isomorphism is the upper floor that we wanna approach. The theories we used for this proof are like the tools and bricks. With them, we are able to build up a stair to get to the upper floor, find something new and continue our exploration. It is actually a [sic] interesting process when you look through the knowledge you learned and find something more than that. And I believe this is what a mathematician should do: they use the known to explore the unknown, and make the unknown known.

While Cedric's realization may seem obvious to mathematicians, it is truly an epiphany for an undergraduate student. We believe it shows clear evidence that Cedric has absorbed key threshold concepts in abstract algebra. In describing the first isomorphism as "the floor we are standing on," and the theories as "tools and bricks" with which to build a stair, Cedric is demonstrating that he has come to see the isomorphism not simply as a process, but a conceptual object that becomes part of his understanding and facilitates deeper mathematical thinking. This is exactly the point of normal discourse: the very process of communicating with other learned professionals in our disciplines, testing ideas and confirming what we know, helps in our attempts to understand what we do not yet know.

Appendix: Writing Associate Training Curriculum (A Brief Outline)

Writing associates at Lafayette College begin each year of service with a day-long orientation program and attend weekly staff meetings throughout the year. At some institutions, this program of training and professional development may be offered as a credit-bearing course, but at Lafayette it is merely a condition of the writing associate's employment, for which they receive a modest stipend. Below is an outline of the topics covered each year, accompanied by a typical resource that associates are asked to read and be prepared to discuss.[2]

Topic	Sample Resources
Understanding student difficulties in academic writing	Bartholomae, "Inventing the University"
Heuristic approaches to tutoring writing	Harris, *Rewriting: How to Do Things with Texts*

The transition to college writing	Jenkins, "Accordions, Frogs, and the 5-Paragraph Theme"
Approaches to tutoring	Brooks, "Minimalist Tutoring: Making the Student Do All the Work" and Shamoon, "A Critique of Pure Tutoring"
Understanding student writing processes	Flower, "Writer-Based Prose"
Supporting English Language Learners	Matsuda and Cox, "Reading an ESL Writer's Text"
Advanced techniques	Elbow, "The Believing Game—Methodological Believing"
Being a reflective practitioner	Murray, "The Listening Eye"

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Notes

[1] J. Christian Tatu has left higher education to work in public education.

[2] Before moving on from the College Writing Program at Lafayette, Christian had begun the process of incorporating anti-racist pedagogy into the writing associate training curriculum. In his final year on the job, that important topic was becoming an integral part of nearly every staff meeting and training session he held.

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